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Growth of entire functions

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THE GROWTH OF ENTIRE FUNCTIONS

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TABLE OF CONTENTS

INTRODUCTION

An entire function is a complex valued function single valued and holomorphic in the finite complex plane. It has a Taylor series representation which has an infinite radius of convergence. There are three different ways an entire function $f(z)$ can behave at infinity:

1. $f(z)$ can have a point of holomorphy at infinity; in this case, $f(z)$ is a constant by the theorem of Liouville.
2. $f(z)$ can have a pole of order greater than or equal to one at infinity, and then $f(z)$ reduces to a polynomial.
3. $f(z)$ can have an essential singularity at infinity, and then $f(z)$ is said to be a transcendental entire function.

Henceforth we will be concerned primarily with the third type of entire functions and the study of their growth as z becomes large.

Since entire functions can be expressed as convergent Taylor series, they possess properties which are similar to those of polynomials. The rate of growth of a polynomial as z becomes large depends on the degree of the polynomial and thus on the number of roots of the polynomial. As we shall see later, the idea of the connection between the growth of a polynomial and the number of zeros it possesses has a generalization in the study of the growth of arbitrary transcendental entire functions.

It is necessary to introduce a growth scale to characterize the growth of an entire function. To this end we define the maximum modulus function for an entire function:

$$M_f(r) = \max_{|z|=r} |f(z)|$$

$M_f(r)$ is an increasing function by the Maximum Modulus Principle and Liouville's theorem.

What we would like to have now is an elementary function with which to compare $M_f(r)$. In our search for such a function, we prove the following lemma.

<u>Lemma 1</u>: Suppose f(z) is an entire transcendental function. If there exists a positive integer n such that $\varliminf_{r \to \infty} \frac{M_f(r)}{r^n} = u < \infty$, then f(z) is a polynomial of degree at most n.

<u>Proof</u>: Let f(z) be expressed in its Taylor series, $f(z) = \sum_{i=1}^{\infty} a_i z^i$, and define $P_n(z)$ by $P_n(z) = \sum_{i=1}^{\infty} a_i z^i$. Define the entire function $\varphi(z)$ by $\varphi(z) = [f(z) - P_n(z)]z^{-n-1}$.

Let $\varepsilon > 0$ be a positive real number and let n be fixed. Since $\varliminf_{r \to \infty} \frac{M_f(r)}{r^n} = u < \infty$, we can choose a sequence of positive numbers $r_1, r_2, r_3, \ldots\ldots$ such that $\lim_{k \to \infty} r_k = \infty$ and $\frac{M_f(r_k)}{r_k^n} < u + \varepsilon$ for $k = 1, 2, 3, \ldots\ldots$.

Since $P_n(z)$ is a polynomial of degree n, for some positive constant B, there exists a positive constant K_B depending on B such that for $|z| = r \geq K_B$, $|\frac{P_n(z)}{z^n}| < B$. Also,

$$|\varphi(z)| = |\frac{f(z) - P_n(z)}{z^{n+1}}| \leq \frac{1}{|z|} [|\frac{f(z)}{z^n}| + |\frac{P_n(z)}{z^n}|].$$

Hence for $|z| = r_k \geq K_B$ we have

$$|\varphi(z)| \leq \frac{1}{r_k} [\frac{M_f(r_k)}{r_k^n} + |\frac{P_n(z)}{z^n}|] < \frac{1}{r_k} [u + \varepsilon + B] = \frac{C}{r_k}$$

where $C = u + \varepsilon + B$. For any z, and $r_k \geq z$ we have, by the Maximum Modulus Principle, $|\varphi(z)| < \frac{C}{r_k}$. Since r_k may be taken arbitrarily large, $|\varphi(z)| = 0$. Thus $f(z) = P_n(z)$.

CHAPTER I

To find a suitable elementary function to use as a comparison function for $M_f(r)$, we seek a function that grows more rapidly than any fixed positive power of r. The function with this property which comes most readily to mind is e^x; this suggests using exponential functions to measure the growth of $M_f(r)$.

We choose a function of the form e^{r^k} where k is a positive constant. Some definitions concerning the rate of growth of $M_f(r)$ with respect to the comparison function e^{r^k} are in order at this point.

<u>Definition</u>: An entire function f(z) is of <u>finite order</u> if there exists a positive number k such that the inequality

$$M_f(r) < e^{r^k}$$

holds for sufficiently large values of r.

<u>Definition</u>: The greatest lower bound of the numbers k, $\rho = \inf k \geq 0$, for which $M_f(r) < e^{r^k}$ holds for all sufficiently large r, is called the <u>order</u> of f(z) and will be denoted by ρ.

Suppose the order of the entire function f(z) is ρ. Then according to the definition of order, for any positive ε there exists R_o depending on ε such that $M_f(r) < e^{r^{\rho+\varepsilon}}$ for $r > R_o$. From the properties of greatest lower bounds, there exists a sequence $\{r_n\}$, $\lim_{r \to \infty} r_n = \infty$, such that

$$M_f(r_n) > e^{r_n^{\rho-\varepsilon}} .$$

Taking the logarithm of both sides of the two previous inequalities twice and dividing by ln r, we have

$$\frac{\ln \ln M_f(r)}{\ln r} < (\rho + \varepsilon) \quad \text{for } r > R_o,$$

and $$\frac{\ln \ln M_f(r_n)}{\ln r_n} > (\rho - \varepsilon) \text{ for large } r_n.$$

But this is precisely what is meant by

$$\rho = \overline{\lim_{r \to \infty}} \frac{\ln \ln M_f(r)}{\ln r}. \qquad (1.1)$$

The preceding result shall be stated as a lemma.

<u>Lemma</u> 2: The order ρ of an entire function is given by formula (1.1).

For entire functions of a given order we can characterize the growth of the maximum modulus function more precisely by introducing the <u>type</u> of the function.

<u>Definition</u>: The <u>type</u> of an entire function f(z) of a given order ρ is the greatest lower bound of the numbers A, $\sigma_f = \inf A \geq 0$, such that $M_f(r) < e^{Ar^\rho}$ holds for all sufficiently large values of r. We shall denote the type of an entire function f(z) by σ_f.

As in the case with order, we can obtain a formula expressing the type of an entire function in terms of the maximum modulus function.

<u>Lemma</u> 3: The type σ_f of an entire function of a given

order ρ is given by the formula

$$\sigma_f = \overline{\lim_{r \to \infty}} \frac{\ln M_f(r)}{r^\rho}. \tag{1.2}$$

<u>Proof</u>: Given $\varepsilon > 0$, there exists a positive real number R_o depending on ε such that $M_f(r) < e^{(\sigma_f+\varepsilon)r^\rho}$ for $r > R_o$. There also exists a sequence $\{r_n\}$, $\lim_{r \to \infty} r_n = \infty$, such that $M_f(r_n) > e^{(\sigma_f-\varepsilon)r_n^\rho}$ for large r_n. Taking the logarithm of both sides and dividing by r^ρ and r_n^ρ respectively in the two previous inequalities, we have

$$\frac{\ln M_f(r)}{r^\rho} < \sigma_f + \varepsilon \quad \text{for } r > R_o,$$

and

$$\frac{\ln M_f(r_n)}{r_n} > \sigma_f - \varepsilon \quad \text{for large } r_n.$$

But this is exactly what is meant by

$$\sigma_f = \overline{\lim_{r \to \infty}} \frac{\ln M_f(r)}{r^\rho}.$$

A function is said to be of <u>maximal</u> type if $\sigma_f = \infty$, of <u>normal</u> type if $0 < \sigma_f < \infty$, and of <u>minimal</u> type if $\sigma_f = 0$.

We shall now examine several examples exhibiting entire functions of different order and type using the formulas previously derived.

<u>Example</u> 1: Consider the entire function

$$f(z) = \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

We shall show that sin z has order $\rho = 1$ and type $\sigma_f = 1$. First we verify an inequality that we need.

$$\sin(x+iy) = \sin x \cos iy + \cos x \sin iy$$
$$= \sin x \cosh y + i \cos x \sinh y.$$

Therefore, $|\sin z| = \sqrt{(\sin x \cosh y)^2 + (\cos x \sinh y)^2}$
$$= \sqrt{\sin^2 x(1+\sinh^2 y) + \cos^2 x \sinh^2 y}$$
$$= \sqrt{\sinh^2 y + \sin^2 x}.$$

Hence we have the following estimate for $|\sin z|$.

$$|\sinh y| = \sqrt{\sinh^2 y} \le |\sin z| \le \sqrt{\sinh^2 y + 1} = \cosh y.$$

This estimate gives us

$$\sinh r \le \max_{|z|=r} |\sin z| \le \cosh r.$$

Substituting the exponential expressions for cosh r and sinh r, we have

$$\frac{e^r - 1}{2} < \frac{e^r - e^{-r}}{2} \le M_f(r) = \max_{|z|=r} |\sin z| \le \frac{e^r + e^{-r}}{2} < \frac{e^r + 1}{2}$$

Thus the order ρ and type σ_f of $f(z) = \sin z$ are both equal to 1.

Example 2: Using an argument similar to the one above, it is easily seen that the function $f(z) = \cos\sqrt{z}$ is of order 1/2 and type 1.

Example 3: We shall show that the function $f(z) = e^{z \ln z}$ is of order 1 and maximal type.

$$\max|f(z)| = \max|e^{z \ln z}| = \max|e^{r(\cos\theta + i\sin\theta)(\ln r + i\arg z)}|$$
$$= \max|e^{r(\ln r \cos\theta - \sin\theta \arg z)}|$$
$$= \max|\exp[r\ln r(\cos\theta - \frac{\sin\theta \arg z}{\ln r})]| = \exp Cr \ln r$$

where C is the maximum of $\cos\theta - \frac{\sin\theta \arg z}{\ln r}$.

Therefore $M_f(r) = e^{Cr \ln r}$ for all $r > 0$. Taking logarithms and dividing by ln r, we have

$$\frac{\ln \ln M_f(r)}{\ln r} = \frac{\ln \ln e^{Cr \ln r}}{\ln r} = \frac{\ln(Cr \ln r)}{\ln r}$$

$$= \frac{\ln C}{\ln r} + \frac{\ln r}{\ln r} + \frac{\ln \ln r}{\ln r}.$$

Therefore

$$\overline{\lim_{r \to \infty}} \frac{\ln \ln M_f(r)}{\ln r} = \overline{\lim_{r \to \infty}} \left[\frac{\ln C}{\ln r} + 1 + \frac{\ln \ln r}{\ln r}\right] = 1 = \rho.$$

$$\frac{\ln M_f(r)}{r^1} = \frac{\ln e^{Cr \ln r}}{r^1} = \frac{Cr \ln r}{r^1} = C \ln r.$$

$$\overline{\lim_{r \to \infty}} \frac{\ln M_f(r)}{r^1} = \overline{\lim_{r \to \infty}} C \ln r = \infty = \sigma_f.$$

<u>Example</u> 4: Obviously, the function, $f(z) = e^{e^z}$ is of infinite order and maximal type.

It is also possible to obtain formulas analogous to (1.1) and (1.2) expressing the order and type of an entire function in terms of its Taylor coefficients. We shall not include the development of these two formulas in this paper but shall state the results ([4] Markushevich pp. 257, 259).

<u>Theorem 1'</u>: Suppose $f(z)$ is an entire function of order ρ having $f(z) = \sum_{n=0}^{\infty} c_n z^n$ as its Taylor series. Then

$$\rho = \overline{\lim_{n \to \infty}} \frac{\ln n}{\ln \frac{1}{\sqrt[n]{|c_n|}}}.$$

<u>Theorem 2'</u>: Suppose $f(z)$ is an entire function of order σ_f

having $f(z) = \sum_{n=0}^{\infty} c_n z^n$ as its Taylor series. Then the type of $f(z)$ is given by

$$\sigma_f = \frac{1}{e\rho} \overline{\lim_{n \to \infty}} n|c_n|^{\rho/n}.$$

CHAPTER II

As in the introduction, we again draw an analogy between polynomials and entire functions. Over an appropriate field, every polynomial can be written as a product of linear factors. The analogy of this in entire functions is the Weierstrass factorization theorem on the representation of entire functions by infinite products. This theorem is the basis for the study of the connection between the growth of an entire function and the distribution of its zeros in the complex plane.

Theorem 1: (Weierstrass) Suppose $\{a_n\}$ is a sequence of nonzero complex numbers arranged in order of increasing modulus and having no finite limit point. Then there exists an entire function whose zeros coincide with the points $\{a_n\}$. The product

$$\pi(z) = \prod_{n=1}^{\infty} G(z/a_n;\ p_n)$$

satisfies the above conditions, provided the integers p_n are chosen so that the product converges uniformly on compact sets. The most general entire function with zeros at the points $\{a_n\}$ is given by

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} G(z/a_n;\ p_n) \qquad (2.1)$$

where m is the multiplicity of the zero of f(z) at the origin and g(z) is an entire function.

Proof: Choose a sequence of natural numbers $\{p_n\}$ such that the series $\sum_{n=1}^{\infty}[z/a_n]^{p_n+1}$ converges uniformly on all compact subsets of the complex plane. Such a sequence can be chosen since for $|z| < R$ the inequality

$$\left|\frac{z}{a_n}\right| < q < 1$$

holds for all sufficiently large values of n and we can choose $p_n = n$.

We now form the infinite product

$$\pi(z) = \prod_{n=1}^{\infty} G(z/a_n;\ p_n)$$

where $G(z/a_n;\ p_n) = (1 - z/a_n)\exp[z/a_n + \ldots + (1/p_n)(z/a_n)^{p_n}]$
and $G(z/a_n;\ 0) = (1 - z/a_n)$.

We shall show that $\pi(z)$ converges uniformly on each compact set that contains none of the points $\{a_n\}$.

To this end we estimate the quantity $|\ln G(u;\ p)|$ for $|u| < q < 1$ and $|\arg(1 - u)| \leq \pi$. Under these conditions, since $\ln(1 - u) = -u - u^2/2 - \ldots\ldots - u^n/n - \ldots\ldots$, we have $\ln G(u;\ p) = -\dfrac{u^{p+1}}{p+1} - \ldots\ldots\ldots - \dfrac{u^{p+n}}{p+n} - \ldots\ldots$ and we can obtain the following estimate by a comparison with a geometric series:

$$|\ln G(u;\ p)| < \frac{|u|^{p+1}}{1 - |u|} \leq \frac{1}{1 - q}\,|u|^{p+1}.$$

From this estimate, for $|z| \leq R$ and $n > n(q,R)$, we have

$$\ln G(z/a_n;\ p_n) \leq \frac{1}{1 - q}\left|\frac{z}{a_n}\right|^{p_n+1}.$$

This implies that the series

$$\sum_{n=1}^{\infty} \ln G(z/a_n;\ p_n)$$

converges uniformly because of the uniform convergence of the series $\sum_{n=1}^{\infty} |\frac{z}{a_n}|^{p_n + 1}$ in $|z| \leq R$.

Therefore the product $\pi(z)$ also converges uniformly on compact subsets of $|z| \leq R$ that do not contain any of the points $\{a_n\}$. Hence, $\pi(z)$ defines an entire function whose zeros coincide with the points $\{a_n\}$ and the first part of the theorem is proved.

Let $f(z)$ have the same nonzero zeros as $\pi(z)$ and let m be the multiplicity of the zero of $f(z)$ at the origin. Then

$$\varphi(z) = \frac{f(z)}{z^m\ \pi(z)}$$

has no zeros and is entire. Hence $\varphi(z) = e^{g(z)}$ where $g(z)$ is entire, and we have $f(z) = z^m e^{g(z)}\ \pi(z)$, which proves the theorem.

Much more can be said about the function $f(z)$ if we place this added restriction on the sequence $\{a_n\}$: the series

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{\beta}}$$

converges for some $\beta > 0$.

Let p be the smallest positive integer such that

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{p}} + 1$$

converges. This implies that $0 \leq p < \beta$ and the series $\sum_{n=1}^{\infty}(z/a_n)^{p_n + 1}$ will still converge uniformly if we put all the $p_n = p$.

The infinite product

$$\pi(z) = \prod_{n=1}^{\infty} G(z/a_n;\ p)$$

is called the <u>canonical product</u> and the integer p is called the genus of the canonical product. If $g(z)$ in (2.1) is a polynomial of degree q, then the canonical product has finite genus and the <u>genus</u> of $f(z)$ is equal to the larger of p or q. If $g(z)$ is not a polynomial or if

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{\beta}}$$

diverges for all $\beta > 0$, then the genus of $\pi(z)$ is infinite.

All of the zeros of an entire function occur in the canonical product in the Weierstrass representation. This suggests that when trying to establish a connection between the zeros of an entire function and its growth we should look initially for a relationship between the zeros of the canonical product and its growth. What we shall actually establish is a dependence between the growth of the canonical product and the density of the distribution of its zeros. As a measure of this density we introduce the convergence exponent of the sequence $\{a_n\}$.

<u>Definition</u>: Suppose the sequence $\{a_n\}$ is arranged in order of increasing modulus and it has no finite limit point. The <u>convergence exponent</u> λ of this sequence will be the greatest lower bound of the numbers β such that the series

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{\beta}}$$

converges.

The obvious relationship, $p \leq \lambda \leq p + 1$, exists between the convergence exponent and the genus of the canonical product. The series $\sum_{n=1}^{\infty} \frac{1}{|a_n|^{\lambda}}$ may or may not converge depending on the sequence $\{a_n\}$. However, if λ is an integer and $\lambda = p$ the series $\sum_{n=1}^{\infty} \frac{1}{|a_n|^{\lambda}}$ diverges while the series $\sum_{n=1}^{\infty} \frac{1}{|a_n|^{\lambda+1}}$ converges.

The density of the sequence $\{a_n\}$ can be measured more precisely by considering the growth of the monotone increasing integer valued function $n(r)$, defined for each r as the number of zeros of the entire function $f(z)$ in the circle $|z| < r$.

<u>Definition</u>: The <u>order</u> of the function $n(r)$ is given by the formula

$$\alpha = \overline{\lim_{r \to \infty}} \frac{\ln n(r)}{\ln r} \tag{2.2}$$

<u>Definition</u>: By the <u>upper</u> <u>density</u> Δ of the sequence $\{a_n\}$, we mean

$$\Delta = \overline{\lim_{r \to \infty}} \frac{n(r)}{r^{\alpha}} . \tag{2.3}$$

If the limit exists, Δ will be called the <u>density</u>.

The next lemma establishes a connection between the order of $n(r)$ and the convergence exponent of the sequence $\{a_n\}$.

<u>Lemma</u> 4: Suppose we are given a sequence of complex

numbers $\{a_n\}$ such that $a_n \neq 0$ for all n and $\lim_{n \to \infty} |a_n| = \infty$. Then the convergence exponent of λ of the sequence $\{a_n\}$ equals the order α of the corresponding function $n(r)$.

Proof: The proof will be given in two parts.

1) Suppose $\delta > 0$ is arbitrary. Then the series $\sum_{n=1}^{\infty} \frac{1}{|a_n|^{\lambda+\delta}}$ converges and can be written as a Stieltjes integral of the form $\int_0^{\infty} \frac{dn(t)}{t^{\lambda+\delta}}$. Integrating by parts, we have

$$\lim_{r \to \infty} \int_0^r \frac{dn(t)}{t^{\lambda+\delta}} = \lim_{r \to \infty} \left[\frac{n(t)}{t^{\lambda+\delta}}\Big|_0^r + (\lambda + \delta) \int_0^r \frac{n(t)}{t^{\lambda+\delta+1}} dt\right]$$

$$= \lim_{r \to \infty} \left[\frac{n(r)}{r^{\lambda+\delta}} + (\lambda + \delta) \int_0^r \frac{n(t)}{t^{\lambda+\delta+1}} dt\right] \quad (2.4)$$

since $n(r) = 0$ for $0 \leq t \leq |a_1|$. Since the series $\sum_{n=1}^{\infty} \frac{1}{|a_n|^{\lambda+\delta}}$ converges, the integral $\int_0^{\infty} \frac{dn(t)}{t^{\lambda+\delta}}$ converges and both positive terms on the right side of (2.4) are bounded. Since the integral $\int_0^r \frac{n(t)}{t^{\lambda+1+\delta}} dt$ is a bounded monotone increasing function, it also converges. Hence, given $1 > \varepsilon > 0$, there exists $R(\varepsilon) > 0$ such that

$$\varepsilon > (\lambda + \delta) \int_r^{\infty} \frac{n(t)}{t^{\lambda+\delta+1}} dt \geq (\lambda + \delta) n(r) \int_r^{\infty} \frac{dt}{t^{\lambda+\delta+1}} =$$

$(\lambda + \delta) n(r) \frac{1}{(\lambda+\delta) r^{\lambda+\delta}}$ for $r > R(\varepsilon)$. Thus $\varepsilon > \frac{n(r)}{r^{\lambda+\delta}}$ for $r > R(\varepsilon)$ which implies that $\varepsilon\, r^{\lambda+\delta} > n(r)$ for $r > R(\varepsilon)$. Taking logarithms and dividing by ln r, we have

$$\lambda + \delta > \frac{\ln n(r)}{\ln r} - \frac{\ln \varepsilon}{\ln r} = \left[\frac{\ln \frac{n(r)}{\varepsilon}}{\ln r}\right] > \frac{\ln n(r)}{\ln r} \quad \text{for } r > R(\varepsilon).$$

Thus $\lambda + \delta \geq \overline{\lim_{r \to \infty}} \frac{\ln n(r)}{\ln r} = \alpha$, which says the order α of $n(r)$ does not exceed the convergence exponent λ since $\delta > 0$ was arbitrary.

2) Let α be the order of the function $n(t)$ defined by formula (2.2). Then, given $\varepsilon > 0$, there exists $K(\varepsilon) > 0$ such that for $t > K(\varepsilon)$, $n(t) < t^{\alpha+\varepsilon/2}$. Let $\gamma = \alpha + \varepsilon$. Then for $t > K(\varepsilon)$, $n(t) < t^{\gamma-\varepsilon/2}$ and $\frac{n(t)}{t^{\gamma+1}} < \frac{1}{t^{\varepsilon/2+1}}$.

Integrating and taking the limit as $r \to \infty$, we have

$$\lim_{r \to \infty} \int_r^\infty \frac{n(t)}{t^{\gamma+1}}\, dt < \lim_{K(\varepsilon) < r \to \infty} \int_r^\infty \frac{1}{t^{\varepsilon/2+1}}\, dt$$

$$= \lim_{K(\varepsilon) < r \to \infty} \frac{(-1 - \varepsilon/2)}{r^{\varepsilon/2}} = 0.$$

Thus the integral $\int_0^\infty \frac{n(t)}{t^{\gamma+1}}\, dt$ converges for $\gamma = \alpha + \varepsilon$.

This implies, from part 1), that $\lim_{r \to \infty} \frac{n(r)}{r^\gamma} = 0$. But

$$\int_0^r \frac{dn(t)}{t^\gamma} = \frac{n(r)}{r^\gamma} + \gamma \int_0^r \frac{n(t)}{t^{\gamma+1}}\, dt.$$

Therefore the integral $\int_0^\infty \frac{dn(t)}{t^\gamma}$ converges, which implies that the series $\sum_{n=1}^{\infty} \frac{1}{|a_n|^\gamma}$ converges. Thus the convergence exponent λ is not greater than the order α of $n(r)$ since $\lambda < \gamma = \alpha + \varepsilon$ for all $\varepsilon > 0$.

Together, parts 1) and 2) of the proof say that $\lambda = \alpha$ and the lemma is proved.

We now turn to the problem of establishing a connection between the order of the canonical product and the convergence exponent of the sequence $\{a_n\}$. To this end we prove a theorem by Borel. However, we shall prove two lemmas before proceeding to the theorem.

<u>Lemma</u> 5: For $p > 0$ and all complex numbers u,

$$\ln|G(u;\ p)| < A_p \frac{|u|^{p+1}}{1 + |u|}$$

where $A_p = 3e(2 + \ln p)$. For $p = 0$

$$\ln|G(u;\ 0)| \leq \ln(1 + |u|).$$

<u>Proof</u>: Recall that $G(u;\ p) = (1 - u)\exp[u + u^2/2 + \ldots + u^p/p]$, and $G(u;\ 0) = (1 - u)$ for $p = 0$. Thus $|G(u;\ 0)| = |1 - u| \leq 1 + |u|$, which implies that $\ln|G(u;\ 0)| \leq \ln(1 + |u|)$ as asserted in the second part of the lemma.

Suppose $p > 0$ and $|u| \leq p/p + 1$. Then

$$\ln|G(u;\ p)| = \operatorname{Re}\{\ln(1 - u) + u + u^2/2 + \ldots + u^p/p\}.$$

Since $\ln(1 - u) = -u - u^2/2 - \ldots - u^n/n - \ldots$,

we have

$$\ln|G(u;\ p)| = \operatorname{Re} \sum_{k=p+1}^{\infty} -\frac{u^k}{k} \leq \sum_{k=p+1}^{\infty} \frac{|u|^k}{k} \leq \frac{|u|^{p+1}}{p + 1} \sum_{k=0}^{\infty} |u|^k$$

$$= \frac{|u|^{p+1}}{(p + 1)(1 - |u|)}.$$

$|u| \leq p/p + 1$ implies that $(p + 1)(1 - u) \geq 1$. Therefore,

$$\ln|G(u;\ p)| \leq \frac{|u|^{p+1}}{(p + 1)(1 - |u|)} \leq |u|^{p+1}.$$

But $0 \leq |u| \leq p/p + 1 < 1$ also implies that $3/1 + |u| > 1$. Also $(2 + \ln p) \geq 2$ since p is an integer and $p > 0$. Hence

$\frac{3(2 + \ln p)}{1 + |u|} = \frac{A_p}{1 + |u|} > 1$, which yields

$\ln|G(u;\ p)| < A_p \frac{|u|^{p+1}}{1 + |u|}$.

Suppose $p > 0$ and $|u| > p/p + 1$. Using the inequality $\ln(1 + |u|) \leq |u|$, and the definition of $G(u;\ p)$ we have

$$\ln|G(u;\ p)| \leq \ln(1 + |u|) + \sum_{k=1}^{p} \frac{u^k}{k} \leq 2|u| + \sum_{k=2}^{p} \frac{u^k}{k}$$

$$= |u|^p \left[\frac{2}{|u|^{p-1}} + \sum_{k=2}^{p} \frac{1}{k|u|^{p-k}}\right].$$ But $|u| > p/p + 1$,

hence $\ln|G(u;\ p)| < |u|^p \left[2(1 + 1/p)^{p-1} + \sum_{k=2}^{p} 1/k(1 + 1/p)^{p-k}\right]$

$$= |u|^p(1 + 1/p)^p \left[2(p/p + 1) + \sum_{k=2}^{p} 1/k(p/p + 1)^k\right]$$

$$< |u|^p(1 + 1/p)^p \left[2 + \sum_{k=2}^{p} 1/k\right]$$ since $p/p + 1 < 1$.

Therefore, $\ln|G(u;\ p)| < |u|^p\ (1 + 1/p)^p \left[2 + \int_1^p \frac{dx}{x}\right]$

$$< |u|^p e\ [2 + \ln p].$$

$|u| > p/p + 1$ implies $3|u|/(1 + |u| > 1$ since p is an integer and $p \geq 1$. We now have that

$$\ln|G(u;\ p)| < [(3|u|/(1 + |u|)]\ |u|^p e(2 + \ln p)$$

$$= A_p \frac{|u|^{p+1}}{1 + |u|}$$

and the lemma is proven.

<u>Lemma 6</u>: If the series $\sum_{n=1}^{\infty} \frac{1}{|a_n|^{p+1}}$ converges, then the cononical product $\pi(z) = \prod_{n=1}^{\infty} G(z/a_n;\ p)$ satisfies the following inequality in the entire complex plane:

$$\ln|\pi(z)| < k_p r^p \left(\int_0^r \frac{n(t)}{t^{p+1}}\,dt + r\int_r^\infty \frac{n(t)}{t^{p+2}}\,dt\right)$$

where $|z| = r$, $k_p = 3e(p + 1)(2 + \ln p)$ for $p > 0$, and $k_o = 1$

<u>Proof</u>:

$$\ln|\pi(z)| = \mathrm{Re}\{\ln \pi(z)\} = \mathrm{Re}\{\sum_{n=1}^{\infty} \ln G(z/a_n;\, p)\}$$

$$= \sum_{n=1}^{\infty} \mathrm{Re}\{\ln G(z/a_n;\, p)\} = \sum_{n=1}^{\infty} \ln|G(z/a_n;\, p)|.$$

From Lemma 5 and the estimate for $\ln|G(z/a_n;\, p)|$, we have

$$\ln|\pi(z)| < \sum_{n=1}^{\infty} A_p \frac{|z|^{p+1}}{|a_n|^p(|a_n|+|z|)} = \sum_{n=1}^{\infty} A_p \frac{r^{p+1}}{|a_n|^p(|a_n|+r)}$$

$$= A_p r^{p+1} \sum_{n=1}^{\infty} \frac{1}{|a_n|^p(|a_n|+r)}.$$

The last series can be written as a Stieltjes integral and our inequality assumes the form

$$\ln|\pi(z)| < A_p r^{p+1} \int_0^\infty \frac{dn(t)}{t^p(t + r)}.$$

If we let $u = \dfrac{1}{t^p(t + r)}$, $dv = dn(t)$, and integrate by parts, our inequality becomes

$$\ln|\pi(z)| < A_p r^{p+1} \left[\frac{n(t)}{t^{p+1}}\Big|_0^\infty + \int_0^\infty \frac{(p + 1)t + rp}{t^{p+1}(t + r)^2}\, n(t)\, dt\right]$$

From Lemma 5, if the series $\sum_{n=1}^{\infty} \dfrac{1}{|a_n|^{p+1}}$ converges, then

$\lim\limits_{r \to \infty} \dfrac{n(t)}{t^{p+1}} = 0.$ Thus we have the inequality

$$\ln|\pi(z)| < A_p r^{p+1} \int_0^\infty \frac{(p + 1)t + rp}{t^{p+1}(t + r)^2}\, n(t)\, dt$$

$$< A_p r^{p+1} \int_0^\infty \frac{(p + 1)t + (p + 1)r}{t^{p+1}(t + r)^2}\, n(t)\, dt$$

$$= A_p r^{p+1}(p+1)\int_0^\infty \frac{n(t)\,dt}{t^{p+1}(t+r)}$$

$$= A_p r^{p+1}(p+1)\left(\int_0^r \frac{n(t)\,dt}{t^{p+1}(t+r)}\right.$$

$$\left. + \int_r^\infty \frac{n(t)\,dt}{t^{p+1}(t+r)}\right)$$

$$\leq A_p r^{p+1}(p+1)\left(\int_0^r \frac{n(t)\,dt}{rt^{p+1}}\right.$$

$$\left. + \int_0^\infty \frac{n(t)\,dt}{t^{p+2}}\right).$$

The last step may be accomplished in the first integral since $t + r > r$, and the replacement of $t + r$ by t in the second integral does not affect the convergence of the integral. Factoring $1/r$ out of both integrals we have the desired inequality. The case where $p = 0$ can be established in a similar manner.

We now prove the theorem of Borel.

<u>Theorem 2</u>: (Borel) The order ρ of the canonical product

$$\pi(z) = \prod_{n=1}^{\infty} G(z/a_n;\ p)$$

does not exceed the convergence exponent λ of the sequence $\{a_n\}$.

<u>Proof</u>: p is the smallest integer such that $\sum_{n=1}^{\infty} \frac{1}{|a_n|^{p+1}}$ converges. Therefore, if λ is the convergence exponent of the sequence $\{a_n\}$, λ satisfies the inequality $p \leq \lambda \leq p + 1$. Suppose that $\lambda < \gamma < p + 1$. Then, since $\lambda = \overline{\lim_{r \to \infty}} \frac{\ln n(t)}{\ln t}$, there exists a constant C_γ such that for $t > 0$, $n(t) < C_\gamma t^\gamma$.

From this inequality and Lemma 6 we have the following estimate.

$$\ln|\pi(z)| < k_p r^p \left[\int_0^r \frac{n(t)}{t^{p+1}}\,dt + r\int_r^\infty \frac{n(t)}{t^{p+2}}\,dt\right]$$

$$< k_p r^p \left[\int_0^r \frac{C_\gamma t^\gamma}{t^{p+1}}\,dt + r\int_r^\infty \frac{C_\gamma t^\gamma}{t^{p+2}}\,dt\right]$$

$$= k_p r^p C_\gamma \left[\int_0^r t^{\gamma-p-1}dt + r\int_r^\infty t^{\gamma-p-2}dt\right]$$

$$= k_p r^p C_\gamma \left[\frac{t^{\gamma-p}}{\gamma-p}\Big|_0^r + \frac{r}{(\gamma-p-1)t^{p+1-\gamma}}\Big|_r^\infty\right]$$

$$= k_p r^p C_\gamma \left[\frac{r^{\gamma-p}}{\gamma-p} + \frac{r}{(\gamma-p-1)r^{p+1-\gamma}}\right]$$

$$= k_p r^\gamma C_\gamma \left[\frac{1}{\gamma-p} + \frac{1}{\gamma-p-1}\right] = C_\gamma B_\gamma r^\gamma$$

where $B_\gamma = k_p\left[\frac{1}{\gamma-p} + \frac{1}{\gamma-p-1}\right]$.

We can replace $\pi(z)$ by $M_\pi(r)$ in the previous estimate and have $\ln M_\pi(r) < C_\gamma B_\gamma r^\gamma$ or $M_\pi(r) < e^{C_\gamma B_\gamma r^\gamma}$. This implies that the order of $\pi(z)$ does not exceed γ and hence, does not exceed the convergence exponent λ since γ is an arbitrary number between λ and $p + 1$.

Suppose $\lambda = p + 1$. Then, from the proof of Lemma 4, $\overline{\lim\limits_{t\to\infty}}\,\frac{n(t)}{t^{p+1}} = 0$ and the integral $\int_0^\infty \frac{n(t)}{t^{p+2}}\,dt$ converges.

Thus, in the inequality

$$\ln|\pi(z)| < k_p r^p \left[\int_0^r \frac{n(t)}{t^{p+1}}\,dt + r\int_r^\infty \frac{n(t)}{t^{p+2}}\,dt\right],$$

the second integral times k_p can be made smaller than any $\varepsilon_2 > 0$ by taking r sufficiently large, say $r > t_o(\varepsilon_2)$. In the first integral, the integrand becomes small for large values of t. Hence, given $\varepsilon_1 > 0$, there is an $r_o(\varepsilon_1)$ such that the integral $k_p \int_0^r \frac{n(t)}{t^{p+1}}\, dt$ is less than $r\varepsilon_1$ for $r > r_o(\varepsilon_1)$. The inequality now becomes

$$\ln|\pi(z)| < r^p(r\varepsilon_1 + r\varepsilon_2) \text{ for } r > \max(r_o(\varepsilon_1), t_o(\varepsilon_2)).$$

Letting $\varepsilon = \varepsilon_1 + \varepsilon_2$, we have

$$\ln|\pi(z)| < \varepsilon r^{p+1}$$

for $r > \max(r_o(\varepsilon_1), t_o(\varepsilon_2))$ which implies that the order of $\pi(z)$ does not exceed $p + 1$ and $\pi(z)$ is of minimal type. Thus for $p < \lambda \leq p + 1$ the order of $\pi(z)$ does not exceed the convergence exponent λ of the sequence $\{a_n\}$ and the theorem is proved.

Corollary: If λ is not an integer and the upper density Δ of the sequence $\{a_n\}$ is finite, then $\pi(z)$ is at most of order λ and normal type; if λ is not an integer and if the density Δ of the sequence $\{a_n\}$ is zero, then $\pi(z)$ is at most of order λ and minimal type.

Proof: If $\Delta = \overline{\lim\limits_{t\to\infty}} \frac{n(t)}{t^\lambda} < \infty$, then given $\varepsilon > 0$ there exists $t_o(\varepsilon)$ such that when $t > t_o(\varepsilon)$, $n(t) < (\Delta + \varepsilon)t^\lambda$. From Lemma 6 we can obtain as in the proof of Borel's Theorem that

$$\ln|\pi(z)| < (\Delta + \varepsilon)B_\lambda r^\lambda$$

where $B_\lambda = k_p[\frac{1}{\lambda - P} + \frac{1}{p + 1 - \lambda}]$. This proves the corollary.

An important thing to note here is that if λ is an integer, we cannot use the above argument to obtain the inequality since λ would equal $p + 1$ and B_λ would not be finite.

Thus far we have established that the order of the canonical product does not exceed the convergence exponent of the sequence $\{a_n\}$. The rest of this chapter will be devoted to obtaining a reverse inequality. This result together with Borel's theorem will say that for canonical products the convergence exponent of the sequence $\{a_n\}$ equals the order of the function.

Before we can prove this assertion we shall need two preliminary results. The first result, Jensen's Theorem, is a special case of the Poisson-Jensen formula ([5] Titchmarsh p. 129). We shall state Jensen's Theorem without proof.

Theorem 3: (Jensen) Suppose $f(z)$ is holomorphic in a circle of radius R with center at the origin and $f(0) \neq 0$. Then

$$\int_0^R \frac{n(t)}{t}\, dt = \frac{1}{2\pi} \int_0^{2\pi} \ln|f(Re^{i\Theta})|\, d\Theta - \ln|f(0)|$$

where $n(t)$ is the number of zeros of $f(z)$ in the circle $|z| < t$.

The next lemma establishes an important estimate for the number of zeros in a circle.

Lemma 7: Suppose $f(z)$ is holomorphic in the circle $|z| \leq er$ and $|f(0)| = 1$. Then $n_f(r) \leq \ln M_f(er)$.

Proof: From the monotonicity of the function $n_f(t)$, we have

$$\int_r^{er} \frac{n_f(t)}{t}\, dt \geq n_f(r) \int_r^{er} \frac{dt}{t} = n_f(r) \ln t \Big|_r^{er} = n_f(r).$$

From Jensen's theorem, we have

$$\int_r^{er} \frac{n_f(t)}{t}\, dt \leq \int_0^{er} \frac{n_f(t)}{t}\, dt = \frac{1}{2\pi} \int_0^{2\pi} \ln|f(ere^{i\Theta})|\, d\Theta.$$

$$\frac{1}{2\pi} \int_0^{2\pi} \ln|f(ere^{i\Theta})|\, d\Theta \leq \frac{1}{2\pi} \int_0^{2\pi} \ln M_f(er)\, d\Theta$$

$$= \frac{\ln M_f(er)}{2\pi}\, \Theta \Big|_0^{2\pi} = \ln M_f(er)$$

since $|f(ere^{i\Theta})| \leq M_f(er)$.

Putting the inequalities together, we have $n_f(r) \leq \ln M_f(er)$ which proves the lemma.

We now have the machinery to prove a converse of Borel's theorem for arbitrary entire functions.

Theorem 4: The convergence exponent of the zeros of an arbitrary entire function does not exceed its order.

Proof: Without loss of generality we can assume that $f(0) = 1$. If not we can define a new function $f_1(z)$, which has the same order and convergence exponent as $f(z)$, by

$$f_1(z) = \frac{m!\, z^{-m}}{f^{(m)}(0)}\, f(z)$$

where m is the order of the zero of $f(z)$ at the origin.

From Lemma 7, $n_f(r) \leq \ln M_f(er)$. Hence,

$$\alpha = \overline{\lim_{r \to \infty}} \frac{\ln n_f(r)}{\ln r} \leq \overline{\lim_{r \to \infty}} \frac{\ln \ln M_f(er)}{\ln er} = \rho$$

where α is the order of the function $n_f(r)$ and ρ is the order of $f(z)$.

Lemma 4 stated that the order α of the function $n_f(r)$ was equal to the convergence exponent λ. Therefore $\alpha = \lambda \leq \rho$ and the theorem is proven.

The final result of this chapter ties together previous results and displays an important relationship between the distribution of the zeros of a canonical product and its type.

Theorem 5: For canonical products the convergence exponent of the sequence $\{a_n\}$ is equal to the order of the canonical product. Furthermore, if the convergence exponent is not an integer, then the canonical product is of maximal, minimal, or normal type according to whether the upper density of the set of zeros $\{a_n\}$,

$$\Delta_f = \overline{\lim_{r \to \infty}} \frac{n_f(r)}{r^\lambda} ,$$

equals infinity, zero, or some finite nonzero number.

Proof: The first statement of the theorem follows immediately from Borel's theorem and Theorem 4.

From the corollary to Borel's theorem we have that $\pi(z)$ is of normal or minimal type if $0 < \Delta_f < \infty$ or $\Delta_f = 0$ respectively if λ is not an integer. From Lemma 7 we have that $n_f(r) \leq \ln M_f(er)$ which implies that

$$\Delta_f = \overline{\lim_{r \to \infty}} \frac{n_f(r)}{r^\lambda} \leq \overline{\lim_{r \to \infty}} e^\lambda \ln \frac{M_f(er)}{(er)^\lambda}$$

But the convergence exponent λ equals the order ρ of the canonical product. Thus $\Delta_f \leq e^\rho \sigma_f$ and if $\Delta_f = \infty$, then $\sigma_f = \infty$ implying that the canonical product is of maximal type, which proves the theorem.

The dependence between the growth of canonical products and the distribution of their zeros has been established in case the convergence exponent λ is not an integer. We now turn to the problem of establishing the same dependence for arbitrary entire functions of non-integral order.

CHAPTER III

In the preceding chapter we investigated the growth of canonical products. Recall that the canonical product is only part of the Weierstrass representation

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} G(z/a_n;\ p)$$

of an arbitrary entire function. The other factors are also entire functions. Hence, to investigate the connection between the growth of an arbitrary entire function and its zeros, we must consider the growth of the product of two entire functions. To accomplish this we must prove several results concerning lower bounds for the modulus of a holomorphic function. The first of these is Caratheodory's inequality for the circle. This inequality exhibits a relationship between the maximum modulus of a function and the maximum modulus of its real part.

Definition: Let $f(z) = u(z) + iv(z)$. The maximum modulus of the real part of $f(z)$, denoted by $A_f(r)$, is defined by

$$A_f(r) = \max_{|z| = r} |u(z)|.$$

Theorem 5: If $f(z)$ is any function holomorphic in the circle $|z| \leq R$, then

$$M_f(r) \leq [A_f(R) - \mathrm{Re}(f(0))] \frac{2r}{R - r} + |f(0)|$$

holds for $|z| \leq r < R$.

Proof: We begin with the formula of Schwarz, ([4] Markushevich, Vol. 2, p. 151)

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(Re^{i\psi}) \frac{Re^{i\psi} + z}{Re^{i\psi} - z} d\psi + iv(0)$$

where $f(z) = u(z) + iv(z)$. This expresses a holomorphic function in the circle $|z| \leq R$ in terms of the boundary values of its real part. By letting $z = 0$ in the formula of Schwarz we get

$u(0) - \frac{1}{2\pi} \int_{-\pi}^{\pi} u(Re^{i\psi})\, d\psi = 0$. Adding this to the right side of Schwarz's formula, we have

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(Re^{i\psi}) \frac{Re^{i\psi} + z}{Re^{i\psi} - z} d\psi + iv(0) + u(0)$$

$$- \frac{1}{2\pi} \int_{-\pi}^{\pi} u(Re^{i\psi})\, d\psi = \frac{1}{2\pi} \int_{-\pi}^{\pi} [u(Re^{i\psi}) \frac{Re^{i\psi} - z}{Re^{i\psi} - z} - u(Re^{i\psi})]$$

$$+ f(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{u(Re^{i\psi})[Re^{i\psi} + z] - u(Re^{i\psi})[Re^{i\psi} - z]}{Re^{i\psi} - z} d\psi$$

$$+ f(0) = \frac{1}{\pi} \int_{-\pi}^{\pi} u(Re^{i\psi}) \frac{z}{Re^{i\psi} - z} d\psi + f(0). \qquad (3.1)$$

If we let $f(z) \equiv 1$, then $\frac{1}{\pi} \int_{-\pi}^{\pi} \frac{z}{Re^{i\psi} - z} d\psi = 0$ results.

If $A_f(R)$ is the maximum modulus of the real part of $f(z)$ in the circle $|z| \leq R$, then for a fixed R, $A_f(R)$ is a constant and

$$\frac{1}{\pi} \int_{-\pi}^{\pi} A_f(R) \frac{z}{Re^{i\psi} - z} d\psi = 0. \qquad (3.2)$$

Subtracting (3.1) and (3.2), we have

$$- f(z) = - \frac{1}{\pi} \int_{-\pi}^{\pi} u(Re^{i\psi}) \frac{z}{Re^{i\psi} - z} d\psi + \frac{1}{\pi} \int_{-\pi}^{\pi} A_f(R) \frac{z}{Re^{i\psi} - z} d\psi - f(0).$$

Combining the two integrals and taking the modulus of both sides we obtain the inequality

$$|f(z)| \leq \frac{1}{\pi}\left| \int_{-\pi}^{\pi} [A_f(R) - u(Re^{i\psi})] \frac{z}{Re^{i\psi} - z} d\psi \right| + |f(0)|$$

or $$|f(z)| \leq \frac{1}{\pi} \int_{-\pi}^{\pi} [A_f(R) - u(Re^{i\psi})] \frac{|z|}{|Re^{i\psi} - z|} d\psi + |f(0)|,$$

since $[A_f(R) - u(Re^{i\psi})]$ is positive.

If $|z| = r < R$, then the least value of $|Re^{i\psi} - z| = |Re^{i\psi} - re^{i\Theta}|$, which would make the integrand as large as possible, occurs when $\psi = \Theta$. This gives us $|Re^{i\psi} - re^{i\Theta}| = |R - r||e^{i\Theta}| = R - r$.

Our inequality now becomes

$$|f(z)| \leq \frac{1}{\pi} \int_{-\pi}^{\pi} [A_f(R) - u(Re^{i\psi})] \frac{r}{R - r} d\psi + |f(0)|$$

$$= \frac{1}{\pi} A_f(R) \frac{r}{R - r} \int_{-\pi}^{\pi} d\psi - \frac{r}{\pi(R - r)} \int_{-\pi}^{\pi} u(Re^{i\psi})\, d\psi$$

$$+ |f(0)| = \frac{1}{\pi} A_f(R) \frac{r}{R - r} 2\pi - \frac{1}{\pi} \frac{r}{R - r} 2\pi\, u(0)$$

$$+ |f(0)| = [A_f(R) - u(0)] \frac{2r}{R - r} + |f(0)|.$$

The above inequality is valid for all $|z| = r < R$; thus, by the maximum modulus principle, we can replace $f(z)$ by $M_f(r)$ and the theorem is proved.

From Theorem 5 we are able to obtain a lower bound for the modulus of a holomorphic function having no zeros in the circle $|z| \leq R$.

<u>Theorem 6</u>: Suppose $f(z)$ is holomorphic in the circle $|z| \leq R$, $f(0) = 1$, and $f(z)$ has no zeros in the circle

$|z| \leq R$. Then the inequality

$$\ln |f(z)| \geq - \frac{2r}{R - r} \ln M_f(R)$$

holds in the circle $|z| \leq r < R$.

<u>Proof</u>: Let $\varphi(z) = \ln f(z)$. $\varphi(z)$ is holomorphic in $|z| \leq R$ since $f(z)$ has no zeros in that circle.

$$\varphi(z) = \ln f(z) = \ln |f(z)| + i\arg f(z).$$

Hence, $A_\varphi(R) = \max\limits_{|z| = R} |\ln|f(z)|| = \ln M_f(R)$.

Applying Caratheodory's inequality for the holomorphic function $\ln f(z)$, we have

$$M_\varphi(r) \leq [\ln M_f(R) - \ln |f(0)|] \frac{2r}{R - r} + |\ln f(0)|,$$

or
$$|\ln f(z)| \leq \frac{2r}{R - r} \ln M_f(R) \quad \text{for } |z| \leq r < R. \quad (3.3)$$

Now,
$$|\ln f(z)|^2 = |\ln|f(z)| + i\arg f(z)|^2 = (\ln|f(z)|)^2 + (\arg f(z))^2 \geq (\ln |f(z)|)^2.$$

This implies that $|\ln f(z)| \geq |\ln|f(z)||$. Hence

$$|\ln f(z)| = |- \ln f(z)| = |\ln 1 - \ln f(z)| = |\ln 1/f(z)| \geq \ln|1/f(z)|.$$

Inequality (3.3) now becomes

$$\frac{2r}{R - r} \ln M_f(R) \geq |\ln f(z)| \geq \ln|1/f(z)| = \ln 1 - \ln|f(z)| = - \ln|f(z)|.$$

Multiplying by $- 1$, we obtain the inequality stated in the theorem.

The preceding theorem is not true in the case where $f(z)$ has zeros in the circle $|z| \leq R$. However, we can obtain a similar estimate if we exclude from the domain of the function a certain set of circles containing the zeros of the function. This estimate will play an important role in our investigation of the growth of the product of two entire functions. Before we can prove this analogous result, we must prove the theorem of Cartan which provides us with a lower estimate for the modulus of a polynomial.

Theorem 7: (Cartan) Given any real number $H > 0$ and comples numbers $\{a_k\}$, $k = 1, 2, \ldots, n$ not necessarily distinct, there is a system of circles in the complex plane with the sum of their radii equal to $2H$ such that for each point z lying outside these circles one has the inequality

$$|z - a_1||z - a_2| \ldots\ldots |z - a_n| > \left(\frac{H}{e}\right)^n.$$

Proof: The proof of this theorem will be given in several stages.

1.) Choose the quantity H/n as the unit of measurement in the complex plane. We shall show the existence of a closed circle in the complex plane that has as many units of distance in its radius as there are points $\{a_k\}$ in the circle.

Form the smallest convex polygon containing all of the points $\{a_k\}$. Choose a point a_j that is a vertex of this convex polygon. Clearly there are circles of arbitrary radius containing only the point a_j. Hence, we can choose a circle containing a_j whose radius length equals the multiplicity of the point a_j, the multiplicity of a_j

being the number of times a_j appears in the sequence $\{a_k\}$.

2.) From the collection of all circles with radius length equal to the number of points interior to the circle, choose the largest. Denote this circle by C_1, where its radius is $\lambda_1 H/n$.

<u>Claim</u>: There is no circle, whose radius is larger than or equal to $\lambda_1 H/n$, containing more points of the set $\{a_k\}$ than the number of units of measurement in its radius.

<u>Proof</u>: We shall prove this claim by contradiction. Suppose there exists a circle B_1 of radius $\lambda H/n$, with $\lambda \geq \lambda_1$, that contains $\lambda' > \lambda$ points of the set $\{a_k\}$. Thus the circle B_2 concentric to B_1 of radius $\lambda' H/n$ contains either λ' or $\lambda'' > \lambda$ points of the set $\{a_k\}$. If B_2 contains λ' points of the set $\{a_k\}$, we have a contradiction of the choice of C_1. If B_2 contains $\lambda'' > \lambda'$ points of the set $\{a_k\}$, construct a circle B_3, concentric to B_2, of radius $\lambda'' H/n$. The circle B_3 then contains λ'' or $\lambda''' > \lambda''$ points of the set $\{a_k\}$. If B_3 contains λ'' points of the set $\{a_k\}$, we have a contradiction as before. If B_3 contains λ''' points of the set $\{a_k\}$, construct a circle B_4 concentric to B_3 with radius $\lambda''' H/n$. Since the set $\{a_k\}$ is finite, a continuation of this process will eventually give us a circle B_m concentric to B_1 of radius $\lambda^* H/n$ such that $\lambda^* > \lambda_1$ and B_m contains λ^* points. This is impossible since C_1 was chosen to be the largest circle having this property, and our claim is proved.

Let A_1 denote the set of points interior to the circle C_1.

3.) Remove the points of the set A_1 from the set $\{a_k\}$. Construct the largest circle containing the same number of points as there are units of measurement in its radius for the set of points $\{a_k\} - A_1$ and denote it by C_2. Let the circle C_2 have radius $\lambda_2 H/n$. We must show that $\lambda_2 \leq \lambda_1$.

Suppose $\lambda_2 > \lambda_1$. If C_2 has the same number of the points $\{a_k\} - A_1$ as there are units of measurement in its radius, then we would have a contradiction of the choice of C_1 since it was the largest circle having this property. If C_2 contains more of the points $\{a_k\} - A_1$ than there are units of measurement in its radius, then the result of part 2) is contradicted. Hence $\lambda_2 \leq \lambda_1$.

Let A_2 denote the set of points interior to the circle C_2.

Now remove the points of the sets A_1 and A_2 from the set $\{a_k\}$. On the remaining points construct the largest circle containing the same number of points as there are units of measurement in its radius and denote it by C_3. Let the circle C_3 have radius $\lambda_3 H/n$ and as before $\lambda_3 \leq \lambda_2 \leq \lambda_1$.

Continuing in this manner we obtain a sequence of circles C_1, C_2, ..., C_p with radii that contain λ_1, λ_2, ..., λ_p units of measurement respectively and satisfying the following inequality: $\lambda_1 \geq \lambda_2 \geq .... \geq \lambda_p$.

Adding the radii of the sequence of circles $\{C_k\}$ we have

$$H/n(\lambda_1 + \lambda_2 + \ldots. + \lambda_p) = H.$$

4.) We now form a set of circles $D_1, D_2, \ldots., D_p$ which are concentric to the circles $C_1, C_2, \ldots., C_p$ but with radii twice as large. Pick a point z exterior to all of the circles $D_1, D_2, \ldots., D_p$. Describe about the point z a circle C_z of radius $\alpha H/n$ where α is some natural number. The circle C_z does not intersect any of the circles C_j that have radii larger than or equal to $\alpha H/n$. Thus, C_z contains only points of the sets A_k where $k \geq \alpha$. From the definition of the sets A_k, we have that after removing all the sets A_k where $k \geq \alpha$, no circle of radius greater than or equal to $\alpha H/n$ can contain as many of the remaining points as there are units of measurement in its radius. Thus C_z can contain at most $\alpha - 1$ points of the set $\{a_k\}$.

5.) We arrange the set $\{a_k\}$ in order of increasing distance from z. Thus, we have that $|z - a_k| > kH/n$ for $k = 1, \ldots., n$ or $|z - a_1||z - a_2| \ldots\ldots |z - a_n|$ $> n!(H/n)^n = (n!/n^n)H^n$.

Stirling's formula states that for large n

$$n! \sim \sqrt{2\pi n}\,(n/e)^n.$$ Dividing both sides by n^n we have that

$$n!/n^n \sim \sqrt{2\pi n}\,1/e^n > 1/e^n \text{ since } \sqrt{2\pi n} > 1.$$

Hence

$$|z - a_1||z - a_2| \ldots.. |z - a_n| > n!(H/n)^n > (H/e)^n$$

which proves the theorem.

We can now exhibit a lower bound for the modulus of a holomorphic function in a domain excluding a set of circles containing the zeros of the function.

Theorem 8: Let $f(z)$ be holomorphic in $|z| \leq 2eR$ $(R > 0)$ with $f(0) = 1$. Let η be an arbitrary positive real number not exceeding $3e/2$. Then inside the circle $|z| \leq R$, but outside a family of circles whose radii have sum $4\eta R$, we have

$$\ln|f(z)| > - H(\eta) \ln M_f(2eR) \text{ where } H(\eta) = 2 + \ln(3e/2\eta).$$

Proof: We construct the function

$$\varphi(z) = \frac{(-2R)^n}{a_1 a_2 \ldots\ldots a_n} \prod_{k=1}^{n} \frac{2R(z - a_k)}{(2R)^2 - \bar{a}_k z}$$

where $a_1, a_2, \ldots\ldots, a_n$ are the zeros of $f(z)$ in the circle $|z| < 2R$. $\varphi(z)$ has no poles since all of the points $\{a_k\}$ are in $|z| < 2R$. Clearly $\varphi(0) = 1$. Also,

$$|\varphi(2Re^{i\theta})| = \left|\frac{(-2R)^n}{a_1 a_2 \ldots\ldots a_n} \prod_{k=1}^{n} \frac{2R(2Re^{i\theta} - a_k)}{(2R)^2 - \bar{a}_k 2Re^{i\theta}}\right|$$

$$= \left|\frac{(-2R)^n}{a_1 a_2 \ldots\ldots a_n}\right| \left| \prod_{k=1}^{n} \frac{2Re^{i\theta} - a_k}{2R - \bar{a}_k e^{i\theta}}\right|$$

$$= \left|\frac{(-2R)^n}{a_1 a_2 \ldots\ldots a_n}\right| \left| \prod_{k=1}^{n} \frac{2Re^{i\theta} - a_k}{(1/e^{i\theta})(2Re^{i\theta} - \bar{a}_k)}\right|$$

$$= \left|\frac{(-2R)^n}{a_1 a_2 \ldots\ldots a_n}\right|.$$

Define a new function $\psi(z)$ by

$$\psi(z) = \frac{f(z)}{\varphi(z)} .$$

$\psi(z)$ has no zeros or poles in the circle $|z| \leq 2R$ since the zeros of $f(z)$ and $\varphi(z)$ occur at the same points. Thus, by Theorem 6, in the circle $|z| \leq 2R$ we have the inequality

$$\ln|\psi(z)| \geq \frac{-2R}{2R - r} \ln M_\psi(2R) \text{ for } |z| \leq R.$$

But $\ln M_\psi(2R) = \ln \max_{|z| = 2R} \left|\frac{f(z)}{\varphi(z)}\right| = \ln \frac{M_f(2R)}{\varphi(2Re^{i\Theta})}$ since $|\varphi(z)|$ is independent of Θ. Using the value of $|\varphi(2Re^{i\Theta})|$ that we computed earlier, we have

$$\ln|\psi(z)| \geq \frac{-2r}{2R - r} [\ln M_f(2R) - \ln \frac{(2R)^n}{|a_1 a_2 \ldots a_n|}] \quad (3.4)$$

for $|z| \leq R < 2R$.

$2[\frac{r}{2R - r}] \leq 2$, since as $r \to R$, $\frac{r}{2R - r}$ approaches 1 from the left. Also, $\frac{(2R)^n}{|a_1 a_2 \ldots a_n|} > 1$ since $|a_i| < 2R$ for $i = 1, \ldots, n$. Hence, $\ln \frac{(2R)^n}{|a_1 a_2 \ldots a_n|}$ is positive and can be omitted from (3.4) without impairing the inequality. Finally we have $\ln|\psi(z)| \geq -2 \ln M_f(2eR)$ (3.5) for $|z| \leq R < 2R$ since $\ln M_f(2eR) \geq \ln M_f(2R)$.

We now turn to the problem of finding a lower estimate for $\varphi(z)$. If $|z| \leq R$, then

$$|(2R)^2 - \bar{a}_k z| \leq |(2R)^2| + |\bar{a}_k z| \leq 4R^2 + 2RR = 6R^2.$$

Therefore, for the denominator of the finite product of $\varphi(z)$, we have this estimate:

$$\prod_{k=1}^{n} |(2R)^2 - \bar{a}_k z| < (6R^2)^n.$$

The numerator of the finite product in $\varphi(z)$ is a polynomial and we can apply Cartan's estimate. Let $H = 2\eta R$ where $3e/2 > \eta > 0$ is arbitrary. Then if z is exterior to a system of circles the sum of whose radii equals $2H = 4\eta R$, we have

$$\left|\prod_{k=1}^{n} 2R(z - a_k)\right| > (2\eta R/e)^n (2R)^n.$$

Therefore $$|\varphi(z)| > \frac{(2R)^n}{|a_1 a_2 \ldots a_n|} (2\eta R/e)^n (2R)^n \frac{1}{(6R^2)^n}$$

$$\geq \frac{(2R)^{2n}}{(2R)^n} (2\eta R/e)^n \frac{1}{(6R^2)^n} = \left(\frac{2\eta}{3e}\right)^n$$

and we have a lower estimate for $\varphi(z)$.

By Lemma 7, $n = n_f(2R) \leq \ln M_f(2eR)$. Hence

$$\ln|\varphi(z)| > \ln\left(\frac{2\eta}{3e}\right)^n = n \ln\left(\frac{2\eta}{3e}\right) = -n \ln\left(\frac{3e}{2\eta}\right)$$

$$\geq - \ln\left(\frac{3e}{2\eta}\right) \ln M_f(2eR) = \ln\left(\frac{2\eta}{3e}\right) \ln M_f(2eR). \quad (3.6)$$

From the definition of $\psi(z)$, we have that

$$\ln|\psi(z)| = \ln\left|\frac{f(z)}{\varphi(z)}\right| = \ln|f(z)| - \ln|\varphi(z)|.$$

Combining this and inequalities (3.5) and (3.6) we have

$\ln|f(z)| - \ln|\varphi(z)| \geq - 2 \ln M_f(2eR)$ which implies that

$$\ln|f(z)| \geq \ln|\varphi(z)| - 2 \ln M_f(2eR) > \ln\left(\frac{2\eta}{3e}\right) \ln M_f(2eR) - 2 \ln M_f(2eR).$$

Simplifying, we have

$\ln|f(z)| > - \left[2 + \ln\left(\frac{3e}{2\eta}\right)\right] \ln M_f(2eR)$ which is the asserted inequality.

When considering the growth of the product of two entire functions of different order, it is easy to show that:

Lemma 8: Suppose $f(z)$ and $g(z)$ are entire and suppose the orders of $f(z)$ and $g(z)$ are ξ and ζ respectively such that $\xi > \zeta$. Then the order of $f(z)g(z)$ will not exceed ξ, the larger of the orders of the factors.

Proof: From the definition of order, given $\varepsilon > 0$, there exists $R_1 > 0$ such that for $r > R_1$, $M_f(r) < e^{Ar^{\xi+\varepsilon}}$ and there exists $R_2 > 0$ such that for $r > R_2$, $M_g(r) < e^{Br^{\zeta+\varepsilon}}$.

Hence, $M_{fg}(r) \leq M_f(r)M_g(r) < e^{Ar^{\xi+\varepsilon} + Br^{\zeta+\varepsilon}} = e^{(A + Br^{\zeta-\xi})r^{\xi+\varepsilon}}$

$$= e^{(A + B/r^{\xi-\zeta})r^{\xi+\varepsilon}} \text{ for } r > \max(R_1, R_2).$$

$\xi - \zeta > 0$ implies $\frac{1}{r^{\xi - \zeta}} \to 0$ as $r \to \infty$. Therefore the order of $f(z)g(z)$ does not exceed ξ, the larger of the orders of the factors.

Using the results proven previously in this chapter, we are able to describe more precisely the growth of the product of two entire functions. This shall be formulated as a theorem.

Theorem 9: (a) The type of the product of two entire functions of different orders is equal to the type of the function having the largest order.

(b) The product of two entire functions of the same order, one of which has normal type σ and the other has minimal type, is an entire function of the same order and normal type.

(c) The product of two entire functions of the same order, one having at most normal type and the other having maximal type, is an entire function of the same order and of maximal type.

<u>Proof</u>: Since the proofs of parts (a), (b), and (c) are similar, we shall prove only part (b).

Suppose $f(z)$ and $g(z)$ are entire functions of order ρ and of normal type σ and minimal type respectively. From the definitions of order and type, given $\varepsilon > 0$ there exist $K_1 > 0$ and $K_2 > 0$ such that

$$M_f(r) < e^{(\sigma+\varepsilon/2)r^\rho} \quad \text{for } r > K_1$$

and

$$M_g(r) < e^{\varepsilon/2r^\rho} \quad \text{for } r > K_2.$$

Therefore

$$M_{fg}(r) \leq M_f(r)M_g(r) < e^{(\sigma+\varepsilon)r^\rho} \quad \text{for } r > \max(K_1, K_2). \tag{3.6}$$

Now we want a lower bound for $M_{fg}(r)$ of the form $e^{(\sigma-\varepsilon)r^\rho}$. To do this we use Theorem 8. We may assume $g(0) = 1$. If not, multiply $g(z)$ by cz^{-n}, the multiplicity of the zero of $g(z)$ at the origin, which does not change the order and type of $g(z)$.

From the definition of type (1.2) and properties of upper limits, given $\varepsilon > 0$ and $\delta > 0$ (but reserving the choice of δ for the moment) there exists $R_1 > 0$, R_1

arbitrarily large, such that

$$M_f(R_1) > e^{(\sigma-\varepsilon/2)R_1^\rho} \tag{3.7}$$

and for all $R \geq R_1$

$$M_g(R) < e^{\delta R^\rho}. \tag{3.8}$$

We can assume $0 < \delta < 1$. Choose $\eta = \delta/8$ and inside the circle $|z| \leq R$ ($R = R_1(1 - \delta)^{-1}$) form the family of excluded circles mentioned in Theorem 8. The sum of their diameters is $8\eta R = 8(\delta/8)R = \delta R$. Thus in the interval (R_1, R) we can find a number r_1 such that the circle $|z| = r_1$ does not intersect any of the excluded circles. Therefore, from Theorem 8, on this circumference we have

$$\ln|g(z)| > -(2 + \ln(3e/2\eta)) \ln M_g(2eR)$$

$$= -(2 + \ln(12e/\delta)) \ln M_g(2eR). \tag{3.9}$$

Since $R_1 < r_1 < R = R_1(1 - \delta)^{-1}$, we have from (3.7)

$$M_f(r_1) > M_f(R_1) > e^{(\sigma-\varepsilon/2)R_1^\rho}.$$

But $R_1 > r_1(1 - \delta)$, this changes the inequality to

$$M_f(r_1) > e^{(\sigma-\varepsilon/2)(1-\delta)^\rho r_1^\rho}. \tag{3.10}$$

The inequalities (3.9) and (3.10) imply that

$$\ln M_{fg}(r_1) > (\sigma-\varepsilon/2)(1-\delta)\, r_1^\rho$$

$$- (2 + \ln(12e/\delta)) \ln M_g(2eR). \tag{3.11}$$

Equation (3.8) implies that

$$\ln M_g(2eR) < \delta(2eR)^\rho \quad \text{for } R \geq R_1.$$

But $\dfrac{r_1}{1 - \delta} > R_1$, hence

$$\ln M_g(2eR) < \delta(2e)^\rho \; (r_1/(1 - \delta))^\rho = \delta(1-\delta)^{-\rho}(2e)^\rho \; r_1^\rho.$$

Substituting this estimate into the inequality (3.11), we have

$$\ln M_{fg}(r_1) > [(\sigma - \varepsilon/2)(1 - \delta)^\rho - (2 + \ln(12e/\delta))\delta(1 - \delta)^{-\rho}(2e)^\rho]r_1^\rho.$$

Given any $\varepsilon > 0$, we can choose $\delta > 0$ so small that the expression in the brackets is not less that $\sigma - \varepsilon$. This is the $\delta > 0$ that we reserved the choice of in expression (3.8).

Our inequality now becomes

$$M_{fg}(r_1) > e^{(\sigma-\varepsilon)r_1^\rho}$$

for a sequence of values $\{r_1\}$ that have no finite limit. This inequality combined with the inequality (3.6), enables us to conclude that

$$\rho = \overline{\lim_{r \to \infty}} \frac{\ln \ln M_{fg}(r)}{\ln r}$$

and

$$\sigma = \overline{\lim_{r \to \infty}} \frac{\ln M_{fg}(r)}{r^\rho}$$

which completes the proof of part (b).

There is one possibility that Theorem 9 does not take into consideration. That is the case where the two entire functions have the same order and type. To fill in this omission we must make a definition, formulate Theorem 9 in terms of this definition, and prove a corollary.

<u>Definition</u>: We shall say that two functions are in the same <u>category</u> if they have the same order and type. If

one function is of larger order it will be in a <u>larger</u> category. If both functions have the same order, then the function having the larger type will belong to the <u>larger</u> category.

<u>Theorem 9'</u>: If two functions are of different categories, their product will have the same order and type as the function belonging to the larger category.

<u>Corollary</u>: If the quotient of two entire functions f(z) and g(z) is an entire function $\psi(z)$, then its category does not exceed the larger of the categories of the functions f(z) and g(z). Here the categories of f(z) and g(z) may be the same. If f(z) and g(z) are of different categories, then the category of $\psi(z)$ equals the larger of the categories of f(z) and g(z).

<u>Proof</u>: $\psi(z) = \frac{f(z)}{g(z)}$ implies that $f(z) = g(z)\psi(z)$.

If the category of $\psi(z)$ is larger than that of g(z), then by Theorem 9 the category of f(z) equals the category of $\psi(z)$. Thus the category of $\psi(z)$ cannot exceed the category of f(z), and the first part of the corollary is proved.

If the category of f(z) exceeds the category of g(z), then by the theorem the category of $\psi(z)$ equals that of f(z). The only way for the category of g(z) to exceed that of f(z) is for the categories of $\psi(z)$ and g(z) to be equal, and we are done.

This theorem and its corollary enables us to determine the category of f(z) in its Weierstrass representation

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} G(z/a_n;p)$$

if we know the categories of $e^{g(z)}$ and the canonical product, providing they are not the same. If their categories are equal, then we know the category of f(z) is less than or equal to the categories of $e^{g(z)}$ and the canonical product.

CHAPTER IV

If we restrict our consideration to entire functions of finite order we can prove that the function g(z) in the Weierstrass representation

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} G(z/a_n:p) \tag{4.1}$$

is a polynomial of degree no larger than the order of f(z). This result is one of the classical theorems of entire functions and is due to Hadamard.

<u>Theorem 10</u>: An entire function f(z) of finite order ρ possesses a representation of the form

$$f(z) = z^m e^{P(z)} \prod_{n=1}^{\omega} G(z/a_n;p) \quad (\omega \leq \infty) \tag{4.2}$$

where the numbers a_n are the zeros of f(z), $p \leq \rho$, P(z) is a polynomial of degree q such that $q \leq \rho$, and m is the multiplicity of the zero of f(z) at the origin.

<u>Proof</u>: By definition p is the smallest integer such that the series $\sum_{n=1}^{\infty} \frac{1}{|a_n|^{p+1}}$ converges. We have already noted the relationship, $p \leq \lambda \leq p + 1$, between the genus of the canonical product and the convergence exponent. Theorem 4 says that the convergence exponent λ of an arbitrary entire function does not exceed its order ρ. Thus for p in the canonical product

$$\pi(z) = \prod_{n=1}^{\omega} G(z/a_n;p) \quad (\omega \leq \infty) \tag{4.3}$$

the relationship $p \leq \lambda \leq \rho$ exists and part of the theorem is proved.

It remains for us to show that $P(z)$ is a polynomial of degree q and that $q \leq p$.

Define a new function $\psi(z)$ by:

$$\psi(z) = \frac{f(z)}{z^m \, \pi(z)} .$$

Recall that the order of the canonical product (4.3) equals the convergence exponent λ of the set of zeros $\{a_k\}$ of $f(z)$ which is less than the order ρ of $f(z)$. Therefore, by the Corollary to Theorem 9, the order of $\psi(z)$ is at most ρ. Also, $\psi(z)$ has no zeros since the zeros of $f(z)$ and $z^m \, \pi(z)$ occur at the same points.

Thus, given $\varepsilon > 0$ there exists $R_o > 0$ such that

$$|\psi(z)| < e^{\sigma R^{\rho+\varepsilon}} \quad \text{for } R > R_o,$$

or
$$\ln|\psi(z)| < \sigma R^{\rho+\varepsilon} \text{ for } R > R_o. \tag{4.4}$$

Since $\psi(z)$ has no zeros, the function $g(z) = \ln\psi(z)$ is entire.

$$|g(z)| = |\ln\psi(z)| = \ln|\psi(z)| + i\arg\psi(z).$$

Hence, $A_g(R) = \ln|\psi(z)|$ and the asymptotic inequality (4.4) implies
$$A_g(R) < \sigma R^{\rho+\varepsilon} \quad \text{for } R > R_o. \tag{4.5}$$

Applying Caratheodory's inequality to $M_g(r)$, we have

$$M_g(r) < [A_g(R) - \text{Re}\{g(0)\}] \frac{2r}{R - r} + |g(0)|$$

for any $0 < r < R$. Let $r = R/2$. Then

$$M_g(r) < [A_g(R) - \text{Re}\{g(0)\}] \frac{2R/2}{R - R/2} + |g(0)|$$

$$= [A_g(R) - \mathrm{Re}\{g(0)\}]\ 2 + |g(0)|$$

$$= 2A_g(R)\ [1 - \frac{\mathrm{Re}\{g(0)\}}{A_g(R)} + \frac{|g(0)|}{2A_g(R)}].$$

$\mathrm{Re}\{g(0)\}$ and $|g(0)|$ are both constants so let them equal C_1 and C_2 respectively. With this substitution and the asymptotic inequality (4.5), we obtain

$$M_g(r) < 2\sigma R^{\rho+\varepsilon}\ [1 - \frac{C_1}{R^{\rho+\varepsilon}} + \frac{C_2}{2R^{\rho+\varepsilon}}] \quad \text{for } R > R_o.$$

For R sufficiently large, say $R > K_o$, the expression in the brackets can be made as close to 1 as we like or, equivalently, less than R^{ε}. The inequality now becomes

$$M_g(r) < 2\sigma R^{\rho+\varepsilon}\ R^{\varepsilon} = 2\sigma R^{\rho+2\varepsilon} \quad \text{for } R > K_o.$$

Thus, by Lemma 1, $g(z)$ is a polynomial of degree at most ρ.

Recall that the genus g of an entire function $f(z)$ equals $\max(p, q)$ where p is the genus of the canonical product and q is the degree of the polynomial $g(z)$. Theorem 10 implies that $g \leq \rho$ where ρ is the order of $f(z)$.

If the order ρ of $f(z)$ is not an integer, then $q < \rho$. By Theorem 9 the order of the canonical product must equal ρ. The order of the canonical product coincides with the convergence exponent λ. The genus p of the canonical product satisfies the inequality $p \leq \lambda \leq p + 1$. Hence $p \leq \rho \leq p + 1$ or $g \leq \rho \leq g + 1$, and the genus g of $f(z)$ equals $[\rho]$ where $[\rho]$ means the largest integer less than or equal to ρ.

Suppose the order ρ of $f(z)$ is an integer. Then from the previous argument g equals ρ or $\rho - 1$. The following lemma gives the necessary and sufficient conditions for $g = \rho - 1$.

Lemma 9: Suppose ρ is an integer. Then the genus g of $f(z)$ equals $\rho - 1$ if and only if λ is an integer, $q < \lambda$, and the series $\sum_{n=1}^{\infty} \frac{1}{|a_n|^{\lambda}}$ converges.

Proof: 1.) Suppose $g = \rho - 1$. The definition of g implies that $g = q$ or $g = p$.

Suppose $g = q$. Then $q = \rho - 1$ or $q + 1 = \rho$. This implies $q < \rho$ and hence ρ coincides with the order of the canonical product which equals the convergence exponent λ by Theorem 5. Therefore λ is an integer and $\lambda = \rho = q + 1 \geq p + 1$. Since the series $\sum_{n=1}^{\infty} \frac{1}{|a_n|^{p+1}}$ converges, $\sum_{n=1}^{\infty} \frac{1}{|a_n|^{\lambda}}$ converges.

Suppose $g = p$. Then $p = \rho - 1$ or $p + 1 = \rho$. This implies that $q \leq p < \rho$. Therefore ρ coincides with the order of the canonical product which equals λ. Thus λ is an integer and $\lambda > q$. The series $\sum_{n=1}^{\infty} \frac{1}{|a_n|^{p+1}}$ converges so the series $\sum_{n=1}^{\infty} \frac{1}{|a_n|^{\lambda}}$ converges since $p + 1 = \rho = \lambda$.

2.) Suppose λ is an integer, $q < \lambda$, and $\sum_{n=1}^{\infty} \frac{1}{|a_n|^{\lambda}}$ converges. λ equals the order of the canonical product by

Theorem 5, and since $\lambda > q$, the order of $f(z)$ must equal λ. Since p is the smallest integer such that $\sum_{n=1}^{\infty} \frac{1}{|a_n|^{p+1}}$ converges and since λ is an integer, we have from the definition of the convergence exponent that $\lambda = p + 1$. Since λ is an integer and $\lambda > q$, we have that $\lambda \geq q + 1$. Thus $\rho = \lambda = p + 1 \geq q + 1$ which implies that $\rho = p + 1 = g + 1$ or $g = \rho - 1$ and the proof is finished.

Some examples are in order at this point.

Example 1: Let $f(z) = \frac{\sin \pi \sqrt{z}}{\pi \sqrt{z}}$. $f(z)$ is entire since $\lim_{z \to 0} f(z) = 1$. We shall show that $f(z)$ has order 1/2 and genus zero. $f(z)$ has a Taylor series representation of the form

$$f(z) = \frac{\sin \pi \sqrt{z}}{\pi \sqrt{z}} = 1 - \frac{(\pi \sqrt{z})^2}{3!} + \frac{(\pi \sqrt{z})^4}{5!} + \cdots$$

$$+ \frac{(-1)^{n+1}(\pi \sqrt{z})^{2n}}{(2n + 1)!} + \cdots.$$

Thus its nth Taylor series coefficient is

$$c_n = \frac{\pi^{2n}}{(2n + 1)!}.$$ From Theorem 1'

the order ρ of $f(z)$ can be expressed as

$$\rho = \overline{\lim_{r \to \infty}} \frac{\ln n}{\ln \frac{1}{\sqrt{\frac{\pi^{2n}}{(2n+1)!}}}} = \overline{\lim_{r \to \infty}} \frac{\ln n}{\ln \left[\frac{(2n + 1)!}{\pi^2}\right]^{1/n}}. \qquad (4.6)$$

Stirling's formula states that $n! \sim \sqrt{2\pi n}\,(n/e)^n$ for sufficiently large n, say $n > K_o$. Substituting this approximation for $(2n + 1)!$ into formula (4.6), we have

$$\rho = \overline{\lim_{K_o<n\to\infty}} \frac{\ln n}{\ln \dfrac{\sqrt[n]{\sqrt{2\pi(2n+1)}}\left(\dfrac{2n+1}{e}\right)^{2+1/n}}{\pi^2}}$$

$$= \overline{\lim_{K_o<n\to\infty}} \frac{\ln n}{\frac{1}{2n}[\ln(2\pi+\ln(2n+1)] + (2+1/n)[\ln(2n+1) - 1] - \ln\pi^2} .$$

Dividing the numerator and denominator by ln n, we have

$$\rho = \overline{\lim_{K_o<n\to\infty}} \frac{1}{\frac{\ln 2\pi}{2n \ln n} + \frac{\ln(2n+1)}{2n \ln n} + \frac{2\ln(2n+1)}{\ln n} + \frac{\ln(2n+1)}{n \ln n} - \frac{2}{\ln n} - \frac{1}{n \ln n} - \frac{\ln \pi^2}{\ln n}}$$

The limit of all the terms of the denominator except $\frac{2\ln(2n+1)}{\ln n}$ are zero. Clearly $\overline{\lim_{K_o<n\to\infty}} \frac{2\ln(2n+1)}{\ln n} = 2.$

Hence the order ρ of $f(z)$ is 1/2.

We have left to show that the genus of $f(z)$ is zero. The zeros of $f(z) = \frac{\sin \pi\sqrt{z}}{\pi\sqrt{z}}$ occur at the points $1^2, 2^2,\ldots,n^2,\ldots$.

The series $\sum_{n=1}^{\infty} \frac{1}{|n^2|^\lambda}$ converges for all $\lambda > 1/2$ and hence the convergence exponent $\lambda = \rho = 1/2$. By Hadamard's Theorem and the remarks following it, the genus of $f(z)$ is zero and $f(z)$ has the representation

$$f(z) = \frac{\sin \pi\sqrt{z}}{\pi\sqrt{z}} = e^0 \prod_{n=1}^{\infty} G(z/n^2;0) = \prod_{n=1}^{\infty}(1 - z/n^2).$$

<u>Example 2</u>. Replacing z by z^2 in example 1 and taking the

proper principal value of z^2, we have that

$$\sin \pi z = \pi z \prod_{n=1}^{\infty}(1 - z^2/n^2) = \pi z \prod_{n=1}^{\infty}(1 - z/n)(1 + z/n)$$

$$= \pi z \prod_{n=1}^{\infty}(1 - z/n)e^{-z/n}(1 + z/n)e^{z/n}$$

$$= \pi z \prod_{n=1}^{\infty}(1 - z/n)e^{-z/n} \prod_{n=1}^{\infty}(1 + z/n)e^{z/n}.$$

Taking the first product over the negative integers and changing the sign of the terms, we have

$$\sin \pi z = \pi z \prod_{n=-\infty}^{\infty}{}' (1 + z/n)e^{z/n}$$

where the (') indicates that $n \neq 0$.

In example 1 Chapter 1 we found that sin z and hence $\sin \pi z$ has order and type equal to 1. The zeros of $\sin \pi z$ occur at the positive and negative integers. Since the series $\sum_{n=1}^{\infty} \frac{1}{|n|^{\lambda}}$ converges for all $\lambda > 1$, the convergence exponent of the zeros of $\sin \pi z$ is 1. By Hadamard's Theorem the genus of $\sin \pi z$ equals 1.

We have already noted from the remarks following Theorem 10 that if an entire function f(z) is of non-integral order, then its order coincides with the order of the canonical product. From this remark and Theorem 5, we have the following theorem.

Theorem 11: If the order ρ of the entire function f(z) is not an integer, then f(z) is of maximal, minimal, or normal type according to whether $\Delta_f = \infty$, $\Delta_f = 0$, or $0 < \Delta_f < \infty$ where Δ_f is the upper density of the zeros of f(z).

Thus far we have only considered entire functions of non-integral order. We shall look briefly into the more complex problem of the growth of integer-ordered entire functions in the next chapter.

CHAPTER V

Two added difficulties arise when considering entire functions of integral order. The first of these problems arises from the fact that the order of such a function may be determined by the exponential factor in (4.1). Thus the convergence exponent of the zeros of the function may be less than its order or the function may not have any zeros and Theorem 11 is no longer valid.

Another peculiarity of a function of integral order is that the type of the function depends not only on the distribution of the zeros but also on the distribution of the arguments of the zeros. This dependence will be illustrated by the following examples.

Example 1. Replacing z by $z^2/2^2$ in Example 1 at the end of Chapter IV, we have

$$\sin \frac{\pi}{2}z = \frac{\pi}{2}z \prod_{n=1}^{\infty} (1 - z^2/(2n)^2).$$

$\sin \frac{\pi}{2}z$ has order 1 and type 1 from Example 1 Chapter I. Note that the zeros of $\sin \frac{\pi}{2}z$ occur at the even integers.

Example 2. Consider the function f(z) defined by

$$f(z) = \frac{1}{\Gamma(z)} = e^{\gamma z} \prod_{n=1}^{\infty} (1 + z/n)e^{z/n}$$

where γ is Euler's constant and $\Gamma(z)$ is the gamma function.

It can be verified that $1/\Gamma(z)$ has order one and maximal type ([4] Markushevich pp. 306, 307, 319). The zeros of $1/\Gamma(z)$ occur at the negative integers since $\Gamma(z)$ has poles at the negative integers.

Clearly the densities of the zeros of $1/\Gamma(z)$ and $\sin \pi/2z$ are the same. However, $1/\Gamma(z)$ is of maximal type and $\sin \pi/2z$ is of normal type. Therefore simply knowing the density of the zeros of an entire function of integral order is not enough to determine its type.

Lindelöf proved a theorem which settled the question of the type of an entire function of integral order. Before we proceed to the theorem we introduce a new quantity:

$$\delta_f(R) = \alpha_\rho + \frac{1}{\rho} \sum_{|a_n| \leq R} a_n^{-\rho}. \tag{5.1}$$

In the above expression, ρ is the order of $f(z)$ and α_ρ is the coefficient of z^ρ in Hadamard's expansion of $f(z)$.

Let $\delta_f = \overline{\lim\limits_{R \to \infty}} |\delta_f(R)|$ and $\gamma_f = \max(\delta_f, \Delta_f)$

where Δ_f is the upper density of the set of zeros of $f(z)$. Note that α_ρ in (5.1) may be zero if the degree of the polynomial $P(z)$ in (4.2) is less than the order of $f(z)$.

We should also note the following which we shall need in the proof of the theorem.

Remark: If $\delta_f(R)$ is defined as in (5.1), then

$$\operatorname{Re}\{\delta_f(R)z^\rho\} \leq |\delta_f(R)z^\rho|$$

and equality holds when the arguments of $\delta_f(R)$ and z^ρ cancel each other.

Theorem 12: (Lindelöf) (a) Suppose $f(z)$ is an entire function of integral order $\rho \geq 1$, and $f(z)$ has representation (4.2). Suppose also that the genus p of the canonical product in (4.2) equals the order ρ of $f(z)$. Then $f(z)$ is of minimal, normal, or maximal type according to whether

$$\gamma_f = 0, \quad 0 < \gamma_f < \infty, \quad \text{or} \quad \gamma_f = \infty.$$

(b) If the genus of $f(z)$ is less than its order, then the type of $f(z)$ is equal to the coefficient of z^ρ in the exponential factor $e^{P(z)}$ appearing in the representation (4.2).

<u>Proof</u>: We shall prove part (a) first. Suppose $p = \rho$. Define a new function $f_R(z)$ by

$$f_R(z) = \prod_{|a_n| \leq R} G(z/a_n;\rho-1) \prod_{|a_n|>R} G(z/a_n;\rho).$$

We shall write $f(z)$ in terms of $f_R(z)$.

$$f(z) = z^m e^{P(z)} \prod_{n=1}^{\infty} G(z/a_n;\rho)$$

$$= z^m e^{P(z)} \prod_{|a_n| \leq R} G(z/a_n;\rho) \prod_{|a_n|>R} G(z/a_n;\rho)$$

$$= z^m e^{P(z)} \prod_{|a_n| \leq R} (1 - z/a_n)\exp(z/a_n + \ldots + (z/a_n)^{\rho-1} \cdot \frac{1}{\rho-1} + (z/a_n)^\rho \cdot \frac{1}{\rho}) \cdot \prod_{|a_n|>R} G(z/a_n;\rho)$$

$$= z^m e^{P(z)} \exp[\sum_{|a_n| \leq R} (z/a_n)^\rho \frac{1}{\rho}] \prod_{|a_n| \leq R} (1 - z/a_n)\exp(z/a_n + \ldots + (z/a_n)^{\rho-1} \cdot \frac{1}{\rho-1}) \cdot \prod_{|a_n|>R} G(z/a_n;\rho)$$

$$= z^m e^{P_{\rho-1}(z)} \exp[(\alpha_\rho + \frac{1}{\rho} \sum_{|a_n| \leq R} a_n^{-\rho}) z^\rho] \prod_{|a_n| \leq R} G(z/a_n; \rho-1)$$

$$\cdot \prod_{|a_n| > R} G(z/a_n; \rho) = z^m e^{P_{\rho-1}(z)} e_f^{\delta(R) z^\rho} f_R(z) \quad (5.2)$$

where $P_{\rho-1}(z)$ is a polynomial of degree at most $\rho - 1$.

In order to estimate the growth of $f(z)$ we first estimate the growth of $f_R(z)$. Define a new function $M_R(r)$ by

$$M_R(r) = \max_{0 \leq \Theta \leq 2\pi} |f_R(re^{i\Theta})|.$$

If $\rho > 1$, from Lemma 5 we have the following estimate for $\ln M_R(R)$ on the circle $|z| = R$.

$$\ln M_R(R) < A_\rho \left[\sum_{n=1}^{[R]} \frac{|z/a_n|^\rho}{1 + |z/a_n|} + \sum_{n=[R]+1}^{\infty} \frac{|z/a_n|^{\rho+1}}{1 + |z/a_n|} \right]$$

$$= A_\rho \left[\sum_{n=1}^{[R]} \frac{|z|^\rho}{|a_n|^{\rho-1}(|a_n| + |z|)} \right.$$

$$\left. + \sum_{n=[R]+1}^{\infty} \frac{|z|^{\rho+1}}{|a_n|^\rho(|a_n| + |z|)} \right]$$

$$= A_\rho \left[\sum_{n=1}^{[R]} \frac{R^\rho}{|a_n|^{\rho-1}(|a_n| + R)} \right.$$

$$\left. + \sum_{n=[R]+1}^{\infty} \frac{R^{\rho+1}}{|a_n|^\rho(|a_n| + R)} \right].$$

Both of the above sums can be written as Stieltjes integrals and our inequality assumes the form:

$$\ln M_R(R) < A_\rho \left[R^\rho \int_0^R \frac{dn(t)}{t^{\rho-1}(t + R)} + R^{\rho+1} \int_R^\infty \frac{dn(t)}{t^\rho(t + R)} \right]. \quad (5.3)$$

If we apply integration by parts to both integrals in (5.3), we can put the inequality in a more useful form.

In the first integral let $u = \dfrac{1}{t^{\rho-1}(t + R)}$ and $dv = dn(t)$. In the second integral let $u = \dfrac{1}{t^\rho(t + R)}$ and $dv = dn(t)$. After carrying out the integration, the following estimate for the first integral can be obtained.

$$\int_0^R \frac{dn(t)}{t^{\rho-1}(t + R)} = \frac{n(t)}{t^{\rho-1}(t + R)}\Big|_0^R - \int_0^R \frac{-\rho(t + R) + R}{t^\rho\ (t + R)^2} n(t)dt$$

$$= \frac{n(R)}{R^{\rho-1}(R + R)} + \int_0^R \frac{\rho(t + R) - R}{t^\rho\ (t + R)^2}\, n(t)dt$$

$$\leq \frac{n(R)}{2R^\rho} + \int_0^R \frac{\rho(t + R)\ n(t)}{t^\rho(t + R)^2}dt$$

$$= \frac{n(R)}{2R^\rho} + \int_0^R \frac{n(t)\ dt}{t^\rho\ (t + R)}$$

$$\leq \frac{n(R)}{2R^\rho} + \frac{\rho}{R} \int_0^R \frac{n(t)}{t^\rho}dt \text{ since } t + R \geq R.$$

An estimate for the second integral takes the following form after integration by parts.

$$\lim_{A \to \infty} \int_R^A \frac{dn(t)}{t^\rho(t + R)} = \lim_{A \to \infty} \left[\frac{n(t)}{t^\rho(t + R)}\Big|_R^A - \int_R^A \frac{-\rho(t + R) - t\ n(t)}{t^{\rho+1}(t + R)^2}\, dt \right]$$

$$= \lim_{A \to \infty} \left[\frac{n(A)}{A^{\rho}(A + R)} - \frac{n(R)}{R^{\rho}(R + R)} + \int_R^A \frac{\rho(t + R) + t\ n(t)}{t^{\rho+1}(t + R)^2} dt \right]$$

$$\leq \lim_{A \to \infty} \frac{n(A)}{A^{\rho}(A + R)} + \int_R^{\infty} \frac{\rho(t + R) + t + R}{t^{\rho+1}(t + R)^2}\ n(t)dt$$

since $\frac{n(R)}{2R} \geq 0$ and $R > 0$.

Hence $\int_R^{\infty} \frac{dn(t)}{t^{\rho}(t + R)} \leq \lim_{A \to \infty} \frac{n(A)}{A^{\rho}(A + R)} + (\rho+1) \int_R^{\infty} \frac{n(t)\ dt}{t^{\rho+1}(t + R)}$.

Since $A + R \geq A$, we have that $\frac{n(A)}{A^{\rho}(A + R)} \leq \frac{n(A)}{A^{\rho+1}}$ for all A.

But since the series $\sum_{n=1}^{\infty} \frac{1}{|a_n|^{\rho+1}}$ converges, this series written as a Stieltjes integral of the form $\int_0^{\infty} \frac{dn(t)}{t^{\rho+1}}$ converges. In the proof of Lemma 4 we showed that these conditions imply that $\lim_{A \to \infty} \frac{n(A)}{A^{\rho+1}} = 0$. Hence the estimate for the second integral becomes

$$\int_R^{\infty} \frac{dn(t)}{t^{\rho}(t + R)} \leq (\rho + 1) \int_R^{\infty} \frac{n(t)\ dt}{t^{\rho+1}(t + R)} \leq (\rho + 1) \int_R^{\infty} \frac{n(t)\ dt}{t^{\rho+2}}$$

since $t + R \geq t$ and substituting t for $t + R$ does not affect the convergence of the integral.

Substituting the estimates we found for the integrals into (5.3), inequality (5.3) assumes the following form.

$$\ln M_R(R) < A_{\rho} \left[R^{\rho} \frac{n(R)}{2R^{\rho}} + \rho \frac{R^{\rho}}{R} \int_0^R \frac{n(t)}{t^{\rho}}\ dt + (\rho + 1) \int_R^{\infty} \frac{n(t)}{t^{\rho+2}}\ dt \right]$$

$$= A_\rho\left[\frac{n(R)}{2} + \rho R^{\rho-1}\int_0^R \frac{n(t)}{t^\rho}\,dt + (\rho + 1)\int_R^\infty \frac{n(t)}{t^{\rho+2}}\,dt\right]. \qquad (5.4)$$

The case where $\rho = 1$ can be done similarily with the same result.

If the upper density of the zeros of $f(z)$ is given by $\Delta_f = \overline{\lim_{t\to\infty}} \frac{n(t)}{t^\rho}$, then, given $\varepsilon > 0$, there exists $K_o > 0$ depending on ε such that $\frac{n(t)}{t^\rho} < \Delta_f + \varepsilon$ or $n(t) < (\Delta_f + \varepsilon)t^\rho$ for $t > K_o$. Using this estimate for $n(t)$ in (5.4), we have

$$\ln M_R(R) < A_\rho\left[\frac{(\Delta_f + \varepsilon)R^\rho}{2} + \rho R^{\rho-1}\int_0^R \frac{(\Delta_f + \varepsilon)t^\rho}{t^\rho}\,dt + + (\rho + 1)R^{\rho+1}\int_R^\infty \frac{(\Delta_f + \varepsilon)t^\rho}{t^{\rho+2}}\,dt\right]$$

$$= A_\rho(\Delta_f + \varepsilon)\left[\frac{R^\rho}{2} + \rho R^{\rho-1}\int_0^R dt + (\rho + 1)R^{\rho+1}\int_R^\infty \frac{dt}{t^2}\right]$$

$$= A_\rho(\Delta_f + \varepsilon)[\frac{R^\rho}{2} + \rho R^\rho + (\rho + 1)R^{\rho+1}\cdot\frac{1}{R}]$$

$$= A_\rho(\Delta_f + \varepsilon)R^\rho[2\rho + 3/2] = K_\rho(\Delta_f + \varepsilon)R^\rho \text{ for } R > K_o. \qquad (5.5)$$

Now that we have an estimate for the growth of $f_R(z)$, we return to the problem of estimating the growth of $f(z)$. From (5.2) we have that

$$|f(z)| \le |z^m||e^{P_{\rho-1}(z)}||e^{\delta_f(R)z^\rho}| \max_{0 \le \theta \le 2\pi} |f_R(z)|$$

which implies that for $|z| = R$

$$M_f(R) \leq R^m \exp[\mathrm{Re}\{P_{\rho-1}(Re^{i\Theta})\}]\exp[\mathrm{Re}\{\delta_f(R)(Re^{i\Theta})\}]M_R(R).$$

Taking the logarithm of both sides, the inequality becomes

$$\ln M_f(R) \leq \ln R^m + \mathrm{Re}\{P_{\rho-1}(Re^{i\Theta})\} + \mathrm{Re}\{\delta_f(R)(Re^{i\Theta})\}$$
$$+ \ln M_R(R).$$

Using the remark preceding Theorem 12 and the fact that $\mathrm{Re}\{P_{\rho-1}(Re^{i\Theta})\}$ grows as $R^{\rho-1}$, the estimate for $\ln M_f(R)$ becomes

$$\ln M_f(R) \leq |\delta_f(R)|\ R^\rho + \ln M_R(R) + O(R^{\rho-1}) + m \ln R \text{ or}$$
$$\frac{\ln M_f(R)}{R^\rho} \leq |\delta_f(R)| + \frac{\ln M_R(R)}{R^\rho} + \frac{O(R^{\rho-1})}{R^\rho} + \frac{m \ln R}{R^\rho}.$$

But both $\frac{O(R^{\rho-1})}{R^\rho}$ and $\frac{m \ln R}{R^\rho}$ behave as $O(1/R)$. This together with the asymptotic inequality (5.5) gives us

$$\frac{\ln M_f(R)}{R^\rho} < |\delta_f(R)| + K_\rho(\Delta_f + \varepsilon) + O(1/R) \quad \text{for } R > K_o.$$

Taking the limit superior of both sides gives us the following upper estimate for the type σ_f of $f(z)$.

$$\sigma_f = \overline{\lim_{R \to \infty}} \frac{\ln M_f(R)}{R^\rho} \leq \overline{\lim_{R \to \infty}} |\delta_f(R)| + K_\rho \Delta_f = \delta_f$$
$$+ K_\rho \Delta_f \leq \gamma_f(1 + K_\rho). \tag{5.6}$$

We shall now establish a reverse inequality to (5.6) We may assume that $f(0) = 1$. Otherwise in place of $f(z)$ one considers $\hat{f}(z) = \frac{m!f(z)}{z^m f^{(m)}(0)}$ which has the same order and type as $f(z)$. Therefore from Lemma 7 we can say that $n(r) \leq \ln M_f(er)$ for every $r \geq 0$

since f(z) is holomorphic in the whole finite complex plane. Dividing by r^{ρ} and taking the limit superior of both sides, we obtain $\overline{\lim}_{r \to \infty} \frac{n(r)}{r^{\rho}} \leq \overline{\lim}_{r \to \infty} e^{\rho} \frac{\ln M_f(er)}{(er)^{\rho}}$ which implies

$$\Delta_f \leq e^{\rho}\sigma_f. \tag{5.7}$$

We are left with the task of finding a similar inequality for δ_f. To this end we estimate f(z) from below. The first step is to find a lower estimate for $f_R(z)$.

$f_R(z)$ satisfies the conditions of Theorem 8 since $f_R(0) = 1$, and it is holomorphic in the finite complex plane. Choose $\eta = 3/2e$. Then inside the circle $|z| \leq r = R/2e$ but outside a family of excluded circles, the sum of whose diameters is less than $4\eta r = 6re^{-3} = 3Re^{-4}$, we have

$$\ln|f_R(z)| > H(\eta) \ln M_R(R) = -6 \ln M_R(R) \tag{5.8}$$

since $H(\eta) = 2 + \ln \frac{3e}{2\eta} = 2 + \ln e^4 = 6$

But $6re^{-3} < r/2$. Hence, there exists $r_1 > 0$ lying between r/2 and r such that the circumference $|z| = r_1$ does not intersect any of the excluded circles. The validity of this statement lies in the fact that the sum of the diameters in the most extreme case (the arguments of the centers of all the excluded circles being equal and their circumferences tangent) is less than the radius of the circle $|z| = r/2$.

Hence from representation (5.2), we see that on the circumference $|z| = r_1$ we have

$$M_f(r_1) \geq |r_1^m||e^{P_{\rho-1}(z)}||e^{\delta_f(R)z^\rho}||f_R(r_1)|$$

$$= r_1^m \exp[\mathrm{Re}\{P_{\rho-1}(z)\}]\exp[\mathrm{Re}\{\delta_f(R)z^\rho\}]|f_R(r_1)|$$

or

$$\ln M_f(r_1) \geq m \ln r_1 + \mathrm{Re}\{P_{\rho-1}(z)\} + \mathrm{Re}\{\delta_f(R)z^\rho\} + \ln|f_R(r_1)|. \quad (5.9)$$

Since $r_1 < r < R$ and since the circumference $|z| = r_1$ does not intersect any of the excluded circles, the inequality (5.8) is valid for $|z| = r_1$ and we may replace $\ln|f_R(r_1)|$ by $-6 \ln M_R(R)$ in (5.9). Since $\mathrm{Re}\{P_{\rho-1}(z)\}$ grows as $r_1^{\rho-1}$ for $|z| = r_1$, we may replace it by $O(r_1^{\rho-1})$. Hence,

$$\ln M_f(r_1) > m \ln r_1 + \mathrm{Re}\{\delta_f(R)z^\rho\} - 6 \ln M_R(R) + O(r_1^{\rho-1}).$$

Transposing, we have

$$\mathrm{Re}\{\delta_f(R)z^\rho\} < \ln M_f(r_1) - m \ln r_1 + O(r_1^{\rho-1}) + 6 \ln M_R(R). \quad (5.10)$$

We choose the argument of z so that the arguments of $\delta_f(R)$ and z^ρ cancel each other. Then equality holds in the expression $\mathrm{Re}\{\delta_f(R)z^\rho\} \leq |\delta_f(R)z^\rho|$ and the inequality in (5.10) will be preserved on substitution of $|\delta_f(R)z^\rho|$ for $\mathrm{Re}\{\delta_f(R)z^\rho\}$. With this substitution we have

$$|\delta_f(R)z^\rho| = |\delta_f(R)|\, r_1^\rho < \ln M_f(r_1) - m \ln r_1 + O(r_1^{\rho-1}) + 6 \ln M_R(R)$$

and division by r_1^ρ gives us

$$|\delta_f(R)| < \frac{\ln M_f(r_1)}{r_1^\rho} - \frac{m \ln r_1}{r_1^\rho} + \frac{O(r_1^{\rho-1})}{r_1^\rho} + \frac{6 \ln M_R(R)}{r_1^\rho} .$$

Both of the expressions, $\frac{m \ln r_1}{r_1^\rho}$ and $\frac{O(r_1^{\rho-1})}{r_1^\rho}$, grow as $O(1/r_1)$ and consequently may be replaced by it. We also want a replacement for $\frac{6 \ln M_R(R)}{r_1^\rho}$. To this end we note that $R = 2er$ and $r_1 > r/2$ implies $4er_1 > R$. Substituting $4er_1$ for R in the asymptotic inequality (5.5) and multiplying both sides of the inequality by $(4e)^\rho$, we have

$$\frac{\ln M_R(R)}{r_1^\rho} < K_\rho(\Delta_f + \varepsilon)(4e)^\rho \text{ for } 4er_1 > K_o .$$

With these substitutions our estimate for $\delta_f(R)$ takes the form

$$|\delta_f(R)| < \frac{\ln M_f(r_1)}{r_1^\rho} + 6K_\rho(\Delta_f + \varepsilon)(4e)^\rho + O(1/r_1)$$

for $4er_1 > K_o$. By taking the supremum of both sides, we obtain $\delta_f \leq \sigma_f + 6K_\rho \Delta_f(4e)^\rho$. Inequality (5.7) gives us $\delta_f \leq \sigma_f + 6K_\rho e\ \sigma_f(4e)^\rho = C_\rho\sigma_f$ where $C_\rho = (1 + 6K_\rho 4^\rho e^{2\rho})$. From this inequality, inequality (5.7), and the definition of γ_f, we have

$$\gamma_f C_\rho^{-1} \leq \sigma_f .$$

The above estimate together with (5.6) implies that the type σ_f of f(z) depends on γ_f which proves the first part of the theorem.

(b) Suppose $p < \rho$. Then $p + 1 \leq \rho$ and the series $\sum_{n=1}^{\infty} \frac{1}{|a_n|^{\rho}}$ converges. Thus the series written as a Stieltjes integral of the form $\int_0^{\infty} \frac{dn(t)}{t^{\rho}}$ converges. Integrating by parts we have $\int_0^{\infty} \frac{dn(t)}{t^{\rho}} = \frac{n(t)}{t^{\rho}}\Big|_0^{\infty} + \rho \int_0^{\infty} \frac{n(t)}{t^{\rho+1}}\, dt$. Since the integral on the left converges, both terms on the right are bounded. In addition to being bounded, $\int_0^{\infty} \frac{n(t)}{t^{\rho+1}}\, dt$ is monotone increasing and hence converges. We have already verified in the proof of Lemma 4 that under these conditions $\lim_{r \to \infty} \frac{n(r)}{r^{\rho}} = 0$.

From Lemma 6 we have the following estimate for the cononical product $\pi(z)$ in the expression for $f(z)$:

$$\ln|\pi(z)| < K_{\rho} r^{\rho-1}\left(\int_0^r \frac{n(t)}{t^{\rho}}\, dt + r \int_r^{\infty} \frac{n(t)}{t^{\rho+1}}\, dt\right) \qquad (5.1)$$

It remains to show that

$$K_{\rho}\left(\frac{1}{r} \int_0^r \frac{n(t)}{t^{\rho}}\, dt + \int_r^{\infty} \frac{n(t)}{t^{\rho+1}}\, dt\right)$$

is less than an arbitrary $\varepsilon > 0$ for r sufficiently large.

Suppose $\varepsilon > 0$. Since the second integral converges, there exists $R_0 > 0$ such that for $r > R_0$, $K_{\rho} \int_r^{\infty} \frac{n(t)}{t^{\rho+1}} dt < \varepsilon/2$.

We now consider the first integral in (5.1). Since $\rho \geq p + 1$, there exists $\delta \geq 0$ such that $t^{\rho} = t^{p+1+\delta}$

Hence $\frac{K_\rho}{r}\int_\eta^r \frac{n(t)}{t^\rho}\,dt = \frac{K_\rho}{r}\int_\eta^r \frac{n(t)}{t^\rho t^{1+\delta}}\,dt$

$$\leq \frac{K_\rho}{r} \sup_{0\leq t\leq r}\left(\frac{n(t)}{t^\rho}\right)\int_\eta^r \frac{dt}{t}$$

for arbitrary $\eta > 0$. Let $B = K_\rho \sup_{0\leq t\leq r}\left(\frac{n(t)}{t^\rho}\right)$. B is a constant since $\lim_{t\to\infty} \frac{n(t)}{t^\rho} = 0$ and K_ρ is a constant. Therefore

$$\frac{K_\rho}{r}\int_\eta^r \frac{n(t)}{t^\rho}\,dt \leq \begin{cases} \frac{B}{r}(\ln r - \ln \eta) \text{ if } \delta = 0 \\ \frac{B}{r}(-r^{-\delta} + \eta^{-\delta}) \text{ if } \delta > 0. \end{cases}$$

Clearly, in either case there exists $N_o > 0$ such that

$$\frac{K_\rho}{r}\int_0^r \frac{n(t)}{t^\rho}\,dt < \varepsilon/2 \quad \text{for } r > N_o.$$

Combining the last two estimates, (5.11) becomes

$$\ln|\pi(z)| < \varepsilon\, r^\rho \quad \text{for } r > \max(R_o, N_o).$$

The asymptotic inequality above implies that $\pi(z)$ grows no faster than a function of order ρ and minimal type. Thus by part (b) of Theorem 9 the type of $f(z)$ coincides with the type of the exponential factor, (the coefficient of z^ρ) and part (b) is proved.

LIST OF REFERENCES

[1] Boas, Ralph Phillip, Jr. Entire Functions. Academic Press, Inc., New York 1954

[2] Hille, Einar. Analytic Function Theory. Volumes 1 and 2. Blaisdell Publishing Company, Waltham, Mass. 1962

[3] Levin, B. Ja. Distribution of Zeros of Entire Functions. American Mathematical Society, Providence, Rhode Island 1964

[4] Markushevich, A. I. Theory of Functions of a Complex Variable. Vol. 2. Prentice Hall, Inc., Englewood Cliffs, N. J. 1965

[5] Titchmarsh, E. C. The Theory of Functions. Oxford University Press, London 1939